# Reduction of Order

Recall that the general solution of a second order homogeneous linear differential equation

(15.1)
$$L[y] = y'' + p(x)y' + q(x)y = 0$$

is given by

(15.2)
$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where $y_1$ and $y_2$ are any two solutions such that

(15.3)
$$W[y_1, y_2](x) = y_1(x)y_2'(x) - y_1'(x)y_2(x) \neq 0 \quad .$$

In this section we shall assume that we have already found one solution $y_1$ of (15.1) and that we are seeking to find another solution $y_2$ so that we can write down the general solution as in (15.2).

So suppose we have one non-trivial solution $y_1(x)$ of (15.1) and suppose there is another solution of the form

(15.4)
$$y_2(x) = v(x)y_1(x) \quad .$$

Then

(15.5)
$$
\begin{aligned}
W[y_1, y_2] &= y_1 y_2' - y_1' y_2 \\
&= y_1(v'y_1 + vy_1') - y_1'(vy_1) \\
&= (y_1)^2 v' \\
&\neq 0
\end{aligned}
$$

unless $v' = 0$. Thus, any solution we construct by multiplying our given solution $y_1(x)$ by a non-constant function $v(x)$ will give us another linearly independent solution.

The question we now wish to address is: how does one find an appropriate function $v(x)$?

Certainly, we want to choose $v(x)$ so that $y_2(x) = v(x)y_1(x)$ satisfies (15.1). So let us insert $y(x) = v(x)y_1(x)$ into (15.1):

(15.6)
$$
\begin{aligned}
0 &= \tfrac{d^2}{dx^2}(vy_1) + p(x)\tfrac{d}{dx}(vy_1) + q(vy_1) \\
&= v''y_1 + 2v'y_1' + vy_1'' + p(x)v'y_1 + p(x)vy_1' + qvy_1 \\
&= v(y_1'' + p(x)y_1' + q(x)y_1) + v''y_1 + (2y_1' + p(x)y_1)v'
\end{aligned}
$$

The first term vanishes since $y_1$ is a solution of (15.1), so $v(x)$ must satisfy

(15.7)
$$0 = y_1 v'' + (2y_1' + p(x)y_1)v'$$

or

(15.8)
$$v'' + \left(p(x) + \frac{2y_1'}{y_1}\right)v' = 0 \quad .$$

Now set

(15.9)
$$u(x) = v'(x) \quad .$$

Then we have

(15.10)
$$u' + \left(p(x) + \frac{2y_1'(x)}{y_1(x)}\right)u = 0 \quad .$$

This is a first order linear differential equation which we know how to solve. Its general solution is

(15.11)
$$\begin{aligned} u(x) &= C\exp\left[-\int^x \left(p(t) + \tfrac{2y_1'(t)}{y_1(t)}\right)dt\right] \\ &= C\exp\left[-\int^x (p(t))\,dt - 2\int^x \tfrac{y_1'(t)}{y_1(t)}dt\right] \end{aligned}$$

Now note that

(15.12)
$$\frac{d}{dt}\ln[y_1(t)] = \frac{y_1'(t)}{y_1(t)} \quad ,$$

so

(15.13)
$$\begin{aligned} \exp\left[-2\int^x \tfrac{y_1'(t)}{y_1(t)}dy\right] &= \exp\left[-2\int^x \tfrac{d}{dt}\left(\ln[y_1(t)]\right)dt\right] \\ &= \exp\left[-2\ln[y_1(x)]\right] \\ &= \exp\left[\ln\left[(y_1(x))^{-2}\right]\right] \\ &= \tfrac{1}{(y_1(x))^2} \quad . \end{aligned}$$

Thus, (15.11) can be written as

(15.14)
$$u(x) = \frac{C}{(y_1(x))^2}\exp\left[-\int^x p(t)dt\right] \quad .$$

Now recall from (15.9) that $u(x)$ is the derivative of the factor v(x) which we originally sought out to find. So

(15.15)
$$\begin{aligned} v(x) &= \int^x u(t)\,dt + D \\ &= \int^x \left[\tfrac{C}{(y_1(t))^2}\exp\left[-\int^t p(t')dt'\right]\right] + D \end{aligned}$$

It is not too difficult to convince oneself that it is not really necessary to carry along the constants of integration $C$ and $D$. For the constant $D$ can be absorbed into the constant $c_1$ of the general solution $y(x) = c_1 y_1(x) + c_2 y_2(x)$, while the factor $C$ can be absorbed into the constant $c_2$. Thus, without loss of generality, we can take $C = 1$ and $D = 0$. So given one solution $y_1(x)$ of (15.1), a second solution $y_2(x)$ of (15.1) can be formed by computing

(15.16)
$$v(x) = \int^x u(t)\,dt$$

where

(15.17)
$$u(x) = \frac{1}{(y_1(x))^2}\exp\left[-\int^x p(t)dt\right]$$

and then setting

(15.18)
$$y_2(x) = v(x)y_1(x) \quad .$$

The general solution of (15.1) is then

(15.19)
$$y(x) = c_1 y_1(x) + c_2 v(x)y_1(x) \quad .$$

This technique for constructing the general solution from single solution of a second order linear homogeneous differential equation is called **reduction of order**.

For those of you who like nice tidy formulae we can write

(15.20)
$$\boxed{y_2(x) = y_1(x)\int^x \frac{1}{(y_1(s))^2}\exp\left[-\int^s p(t)dt\right]ds}$$

for the second solution.

EXAMPLE 15.1.

(15.21)
$$y_1(x) = e^{-x}$$

is one solution of

(15.22)
$$y'' + 2y' + y = 0 \quad .$$

Find another linearly independent solution.

Well, $p(x) = 2$, so

(15.23)
$$\begin{aligned} u(x) &= \frac{C}{(y_1(x))^2} \exp\left[-\int^x p(t)dt\right] \\ &= \frac{C}{e^{-2x}} \exp[-2x] \\ &= Ce^{2x}e^{-2x} \\ &= C \end{aligned}$$

So

(15.24)
$$v(x) = \int^x u(t)\,dt = \int^x C\,dt = Cx.$$

Thus,

(15.25)
$$y_2(x) = v(x)y_1(x) = Cxe^{-x} \quad .$$

EXAMPLE 15.2.

(15.26)
$$y_1(x) = x$$

is a solution of

(15.27)
$$x^2 y'' + 2xy' - 2y = 0 \quad .$$

Use reduction of order to find the general solution.

Well, we first put the differential equation in standard form:

(15.28)
$$y'' + \frac{2}{x}y' - \frac{2}{x^2}y = 0 \quad .$$

Thus,

(15.29)
$$p(x) = \frac{2}{x} \quad , \quad q(x) = \frac{-2}{x^2} \quad .$$

We first compute $u(x)$.

(15.30)
$$\begin{aligned} u(x) &= \frac{C}{(y_1(x))^2} \exp\left[-\int^x p(t)dt\right] \\ &= \frac{C}{x^2} \exp\left[\int^x -\frac{2}{t}dt\right] \\ &= \frac{C}{x^2} \exp[-2\ln[x]] \\ &= \frac{C}{x^2}x^{-2} \\ &= Cx^{-4} \quad . \end{aligned}$$

So

(15.31)
$$\begin{aligned} v(x) &= \int^x u(t)dt \\ &= \int^x Ct^{-4}dt \\ &= -\frac{C}{3}x^{-3} \end{aligned}$$

and

(15.32)
$$y_2(x) = v(x)y_1(x) = -\frac{C}{3x^3}x = C'x^{-2} \quad .$$

The general solution of the original differential equation is thus

(15.33)
$$y(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 x + c_2 x^{-2} \quad .$$