LECTURE 14

Second Order Linear Equations, General Theory

1. Standard Form

A second order linear differential equation is a differential equation of the form

$$A(x)y'' + B(x)y' + C(x)y = D(x) \quad . \tag{14.1}$$

(Here A, B, C and D are certain prescribed functions of x.)

As in the case of first order linear equations, in any interval where $A(x) \neq 0$, we can replace such an equation by an equivalent one in **standard form**:

$$y'' + p(x)y' + q(x)y = g(x) \tag{14.2}$$

where

$$\begin{aligned} p(x) &= \frac{B(x)}{A(x)} \\ q(x) &= \frac{C(x)}{A(x)} \\ g(x) &= \frac{D(x)}{A(x)} \end{aligned} \tag{14.3}$$

2. Homogeneous vs. Non-homogeneous Linear Differential Equations

In the development that follows it will be important to distinguish between the case when the right hand side of

$$y'' + p(x)y' + q(x)y = g(x) \tag{14.4}$$

is zero or non-zero. We shall say that a second order linear ODE is **homogeneous** if it can be written in the form

$$y'' + p(x)y' + q(x)y = 0 \tag{14.5}$$

otherwise (if $g(x) \neq 0$) we shall say that it is **non-homogeneous**. Note that this terminology is completely unrelated to homogeneous equations of degree zero (the topic of the preceding lecture).

3. Differential Operator Notation

Consider the general second order linear differential equation

$$\phi'' + p(x)\phi' + q(x)\phi = g(x) \quad . \tag{14.6}$$

We shall often write differential equations like this as

$$L[\phi] = g(x) \tag{14.7}$$

where L is the **linear differential operator**

$$L = \frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x) \quad . \tag{14.8}$$

That is to say, L is the operator that acts on a function ϕ by

$$
\begin{aligned}
L[\phi] &= \left(\tfrac{d^2}{dx^2} + p(x)\tfrac{d}{dx} + q(x)\right)\phi \\
&= \tfrac{d^2\phi}{dx^2} + p(x)\tfrac{d\phi}{dx} + q(x)\phi \quad .
\end{aligned} \tag{14.9}
$$

4. General Theorems

The following theorem tells us the conditions for the existence and uniqueness of solutions of a second order linear differential equation.

THEOREM 14.1. *If the functions p, q and g are continuous on an open interval $I \subset \mathbb{R}$ containing the point x_o, then in some interval about x_o there exists a unique solution $y = \phi(x)$ to the differential equation*

$$
y'' + p(x)y' + q(x)y = g(x) \tag{14.10}
$$

satisfying the prescribed initial conditions

$$
\begin{aligned}
y(x_o) &= y_o \\
y'(x_o) &= y'_o \quad .
\end{aligned} \tag{14.11}
$$

Note how this theorem is analogous to the corresponding theorem for first order linear ODE's. Note also that the conditions for existence and uniqueness are fairly lax - all we require is the continuity of the functions p, q, and g around a given initial point. Finally, we note that the form of the initial conditions involves the specification of **both** $y(x)$ and its derivative $y'(x)$ at an initial point x_o.

I should also point out that the preceding theorem does not address the issue of how to construct a solution of a second order linear ODE. Indeed, the actual construction of solutions to second order linear ODE is sufficiently complicated to that we shall spend 90% of the remaining lectures on techniques of solution. The next two theorems at least tell us the basic ingredients for a general solution of a second order linear ODE.

THEOREM 14.2. *(The Superposition Principle) If $y = y_1(x)$ and $y = y_2(x)$ are two solutions of the differential equation*

$$
L[y] = \frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \tag{14.12}
$$

then any linear combination

$$
y = c_1y_1(x) + c_2y_2(x) \tag{14.13}
$$

of $y_1(x)$ and $y_2(x)$, where c_1 and c_2 are constants, is also a solution of (14.12).

Proof.

$$
\begin{aligned}
L\left[c_1y_1 + c_2y_2\right] &= \tfrac{d^2}{dx^2}\left(c_1y_1 + c_2y_2\right) + p(x)\tfrac{d}{dx}\left(c_1y_1 + c_2y_2\right) + q(x)\left(c_1y_1 + c_2y_2\right) \\
&= c_1\left(\tfrac{d^2y_1}{dx^2} + p(x)\tfrac{dy_1}{dx} + q(x)y_1\right) + c_2\left(\tfrac{d^2y_3}{dx^2} + p(x)\tfrac{dy_2}{dx} + q(x)y_2\right) \\
&= c_1 \cdot 0 + c_2 \cdot 0 \\
&= 0
\end{aligned} \tag{14.14}
$$

∎

The fact that a linear combination of solutions of a **linear, homogeneous differential equation** is also a solution is extremely important. The theory of linear homogeneous equations, including differential equations involving higher derivatives depends strongly on the superposition principle.

Example 14.3.

$$y_1(x) = \cos(x) \tag{14.15}$$

and

$$y_2(x) = \sin(x) \tag{14.16}$$

are both solutions of

$$y'' + y = 0. \tag{14.17}$$

It is easy to check that any linear combination of y_1 and y_2 is also a solution.

Example 14.4.

$$y_1(x) = 1 \tag{14.18}$$

and

$$y_2(x) = x^{1/2} \tag{14.19}$$

are both solutions of

$$yy'' + (y')^2 = 0. \tag{14.20}$$

However, it is easy to check that $y_1 + y_2 = 1 + \sqrt{x}$ is not a solution of (14.20). The reason for this lies in the fact that (14.20) is not linear.

Given two solutions y_1 and y_2 of a second order linear homogeneous differential equation

$$L[y] = 0 \quad , \tag{14.21}$$

we can construct an infinite number of other solutions

$$y(x) = c_1y_1(x) + c_2y_2(x) \tag{14.22}$$

by letting c_1 and c_2 run through $\mathbb{R}$. The following question then arises: are **all** the solutions of (14.21) capable of being expressed in form (14.22) for some choice of c_1 and c_2?

This will not always be the case; and so we shall say that two solutions y_1 and y_2 form a **fundamental set** of solutions to (14.21) if every solution of (14.21) can be expressed as a linear combination of y_1 and y_2.

Theorem 14.5. *If p and q are continuous on an open interval $I = (\alpha, \beta)$ and if y_1 and y_2 are solutions of the differential equation*

$$L[y] = y'' + p(x)y' + q(x)y = 0 \tag{14.23}$$

satisfying

$$W(y_1, y_2) = y_1(x)y_2'(x) - y_1'(x)y_2(x) \neq 0 \tag{14.24}$$

at every point $x \in I$, then any other solution of (14.23) on the interval I can be expressed uniquely as a linear combination of y_1 and y_2.

Proof.

Let y_1 and y_2 be two given solutions on an interval I and let Y be an any other solution on I. Choose a point $x_o \in I$. From our basic uniqueness and existence theorem (Theorem 3.2), we know that there is only solution $y(x)$ of (14.23) such that

$$\begin{aligned} y(x_o) &= Y(x_o) \\ y'(x_o) &= Y'(x_o) \quad . \end{aligned} \tag{14.25}$$

namely, $Y(x)$. Therefore if we can show that a solution of the form

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

satisfies the initial conditions (14.25), then we must have $Y(x) = c_1y_1(x) + c_2y_2(x)$ and so $Y(x)$ is a linear combination of $y_1(x)$ and $y_2(x)$.

Thus, we now seek to define constants c_1 and c_2 so that these initial conditions can be matched. We thus set

$$\begin{aligned} c_1 y_1(x_o) + c_2 y_2(x_o) &= y_o \\ c_1 y_1'(x_o) + c_2 y_2'(x_o) &= y_o' \quad . \end{aligned} \tag{14.26}$$

This is just a series of two equations with two unknowns. Solving the first equation for c_1 yields

$$c_1 = \frac{y_o - c_2 y_2(x_o)}{y_1(x_o)} \quad . \tag{14.27}$$

Plugging this into the second equation yields

$$\frac{y_o - c_2 y_2(x_o)}{y_1(x_o)} y_1'(x_o) + c_2 y_2'(x_o) = y_o' \tag{14.28}$$

or

$$y_o y_1'(x_o) - c_2 y_2(x_o) y_1'(x_o) + c_2 y_1(x_o) y_2'(x_o) = y_1(x_o) y_o' \tag{14.29}$$

or

$$c_2 = \frac{y_1(x_o) y_o' - y_1'(x_o) y_o}{y_1(x_o) y_2'(x_o) - y_1'(x_o) y_2(x_o)} \quad . \tag{14.30}$$

Plugging this expression for c_2 into (14.27) yields

$$c_1 = \frac{y_o y_2'(x_o) - y_2(x_o) y_o'}{y_1(x_o) y_2'(x_o) - y_1'(x_o) y_2(x_o)} \quad . \tag{14.31}$$

Thus, we can solve for c_1 and c_2 whenever the denominator

$$W(y_1, y_2) = y_1(x_o) y_2'(x_o) - y_1'(x_o) y_2(x_o) \tag{14.32}$$

does not vanish. Thus, so long as y_1 and y_2 satisfy (14.23) we can always express any solution as a linear combination of y_1 and y_2. ∎

Remark: The quantity

$$W(y_1, y_2) = y_1(x_o) y_2'(x_o) - y_1'(x_o) y_2(x_o) \tag{14.33}$$

is called the **Wronskian** of y_1 and y_2.

Example 14.6. Show that

$$y_1(x) = \cos(x) \tag{14.34}$$

and

$$y_2(x) = \sin(x) \tag{14.35}$$

are form a set of fundamental solutions to the differential equation

$$y'' + y = 0 \quad . \tag{14.36}$$

We simply have to check that the Wronskian does not vanish:

$$\begin{aligned} W(y_1, y_2) &= y_1(x_o) y_2'(x_o) - y_1'(x_o) y_2(x_o) \\ &= \cos(x)\left(\cos(x)\right) - \left(-\sin(x)\right)\sin(x) \\ &= 1 \\ &\neq 0 \quad . \end{aligned} \tag{14.37}$$

Since the Wronskian does not vanish, y_1 and y_2 must be linearly independent.