1. Classify the following differential equations: determine their order, if they are linear or non-linear, and if they are ordinary differential equations or partial differential equations.

(a) \( y'' + \cos(y) = x \)
- 2\(^{nd}\) order, nonlinear, ODE

(b) \( \frac{\partial \Phi}{\partial y} + \frac{\partial^2 \phi}{\partial x^2} = y^2 \)
- 2\(^{nd}\) order, linear, PDE

(c) \( \frac{d^3 x}{dt^3} + x^2 \frac{dx}{dt} + x = 0 \)
- 3\(^{rd}\) order, non-linear, ODE

(d) \( a(x) y' + b(x) y + c(x) = 0 \)
- 1\(^{st}\) order, linear, ODE

(e) \( \frac{dx}{dt} = x^2 \)
- 1\(^{st}\) order, nonlinear, ODE

2. Consider the following first order ODE: \( y' = x + y \) and suppose \( y(x) \) is the solution satisfying \( y(1) = 1 \). Use the numerical (Euler) method with \( n = 3 \) and \( \Delta x = 0.1 \) to estimate \( y(1.3) \).

- We will begin constructing a table of approximate values for points \((x_i, y_i \approx y(x_i))\) on the solution using the Euler formula

\[
\begin{align*}
x_{i+1} &= x_i + \Delta x \\
y_{i+1} &= y_i + F(x_i, y_i) \Delta x
\end{align*}
\]

with \( F(x, y) = x + y \) and \( x_0 = 1, y_0 = 1 \).

\[
\begin{align*}
x_1 &= x_0 + \Delta x = 1.1 \\
y_1 &= y_0 + m(x_0, y_0) \Delta x = y_0 + (x_0 + y_0) \Delta x = 1 + (1 + 1) (0.1) = 1.2 \\
x_2 &= x_1 + \Delta x = 1.2 \\
y_2 &= y_1 + m(x_1, y_1) \Delta x = y_1 + (x_1 + y_1) \Delta x = 1.2 + (1.1 + 1.2) (0.1) = 1.43 \\
x_3 &= x_2 + \Delta x \\
y_3 &= y_2 + m(x_2, y_2) \Delta x = y_2 + (x_2 + y_2) \Delta x = 1.43 + (1.2 + 1.43) (0.1) = 1.693
\end{align*}
\]

So \( y(1.3) \approx 1.693 \).
3. Find an explicit solution of the following (separable) differential equation.
\[ 2x - e^{2y}y' = 0 \]

- We have \( M(x) = 2x \) and \( N(y) = -e^{2y} \), as an implicit solution we’ll have

\[
\int 2x \, dx - \int e^{2y} \, dy = C \quad \Rightarrow \quad x^2 - \frac{1}{2} e^{2y} = C
\]

Solving for \( y \) we obtain

\[
y = \frac{1}{2} \ln |2x^2 - 2C|
\]

\[ \square \]

4. Solve the following initial value problem
\[ y' - \frac{3}{x}y = x \quad , \quad y(1) = 2 \]

- This is a first order linear equation with \( p(x) = -3/x \) and \( g(x) = x \). So the general solution is

\[
\mu(x) = \exp \left( \int \frac{-3}{x} \, dx \right) = \exp \left( -3 \ln |x| \right) = x^{-3}
\]

\[
y(x) = \frac{1}{\mu} \int ug \, dx + \frac{C}{\mu} = \frac{1}{x^{-3}} \int x^{-3}(x) \, dx + \frac{C}{x^{-3}} = x^3 \int x^{-2} \, dx + C x^{-3}
\]

\[
= x^3 \left( -\frac{1}{-1} x^{-1} \right) + C x^3 = -x^2 + C x^3
\]

Plugging the general solution into the initial condition yields

\[
y(1) = \left[ -x^2 + C x^3 \right]_{x=1} = -1 + C \quad \Rightarrow \quad C = 3
\]

\[ \Rightarrow \quad y = -x^2 + 3x^3 \]

\[ \square \]

5. (15 pts) Show that the following equation is exact.
\[ \frac{y}{x} + 2x + \ln |x| \frac{dy}{dx} = 0 \]

and then find the explicit solution of this differential equation.

- For this problem, we have \( M(x, y) = y/x + 2x \) and \( N(x, y) = \ln |y| \). We have

\[
\frac{\partial M}{\partial y} = \frac{1}{x} = \frac{\partial N}{\partial x}
\]

so the equation is exact. Let’s now find an explicit solution to the following initial value problem

\[
\frac{y}{x} + 2x + \ln |x| \frac{dy}{dx} = 0
\]

\[
\Phi(x, y) = \int M \, dx + C_1(y) = \int \left( \frac{y}{x} + 2x \right) \, dx + C_1(y) = y \ln |x| + x^2 + C_1(y)
\]

\[
= \int N \, dy + C_2(x) = \int \ln |x| \, dy + C_2(x) = \ln |x| y + C_2(x)
\]

The consistency for these two expression for \( \Phi \) requires \( C_1(y) = 0 \) and \( C_2(x) = x^2 \). Thus, \( \Phi = y \ln |x| + x^2 \). Our implicit solution is thus

\[
y \ln |x| + x^2 = C \quad \Rightarrow \quad y = \frac{C - x^2}{\ln |x|}
\]

\[ \square \]
6. Due to its radioactivity Carbon 14 decays according to a simple first order linear ODE.

\[ \frac{dQ}{dt} = -kQ \]

(here \( Q(t) \) represents the quantity of Carbon 14 at time \( t \)). If the half-life of Carbon 14 (the time it takes to decay to \( \frac{1}{2} \) quantity) is 5730 years, find \( Q(t) \) for a specimen that originally contained 10 grams of Carbon 14.

- The general solution to the differential equation

\[ \frac{dQ}{dt} + kQ = 0 \]

is readily seen to be (since it is first order linear)

\[ Q(t) = \frac{1}{\exp(\int k \, dt) \exp(\int k \, dt)} \left[ 0 \cdot \exp(\int k \, dt) + \frac{C}{\exp(\int k \, dt)} \right] = 0 + Ce^{-kt} = Ce^{-kt} \]

The constant of integration \( C \) corresponds to the initial quantity at time \( t = 0 \)

\[ 10 = Q(0) = Ce^{-k \cdot 0} \quad \Rightarrow \quad C = 10 \]

To figure out \( k \), we use the half-life information. In 5730 years \( Q(0) \) should be reduced to \( \frac{1}{2} Q(0) \).

Thus,

\[ \frac{1}{2} (10) = 10e^{-5730k} \quad \Rightarrow \quad e^{-5730k} = \frac{1}{2} \]

or

\[ k = -\frac{1}{5730} \ln \left( \frac{1}{2} \right) = \frac{\ln 2}{5730} \]

Thus,

\[ Q(t) = 10 \exp \left( \frac{\ln 2}{5730} \cdot t \right) \]