LECTURE 8

Exact Equations

We shall now present another technique for solving first order, non-linear, ordinary differential equations. This technique is a generalization of the one we used for separable equations. To motivate it, let us consider first the equation

$$\psi(x,y) = C \tag{8.1}$$

which fixes "implicitly" y as a function of x, because, at least in principle we should be able to solve (8.1) for y. On the other hand, we can also associate with (8.1) a differential equation by regarding y as a function of x and differentiating (8.1) with respect to x:

$$\frac{d}{dx}\left(\psi(x,y) = C\right) \qquad \Rightarrow \qquad \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0 \tag{8.2}$$

Here's an explicit example. Consider the equation of a circle:

$$x^2 + y^2 = R^2. \tag{8.3}$$

If we differentiate this equation as in (8.2) we get

$$2x + 2y\frac{dy}{dx} = 0$$

or

$$y' + \frac{x}{y} = 0 \tag{8.4}$$

which is a first order non-linear ODE. Now the (algebraic) solutions of (8.3) are given by

$$y = \pm\sqrt{R^2 - x^2} \quad . \tag{8.5}$$

Note that, as self-consistency demands, the functions

$$y(x) = \pm\sqrt{R^2 - x^2}$$

are also solutions of the differential equation (8.4); e.g.,

$$\frac{d}{dx}\left(\sqrt{R^2 - x^2}\right) + \frac{x}{\sqrt{R^2 - x^2}} = \frac{-x}{\sqrt{R^2 - x^2}} + \frac{x}{\sqrt{R^2 - x^2}} = 0 \quad .$$

Thus, to every algebraic relation of the form (8.1) we have an associated ODE (8.2), and the solutions $y(x)$ of the ODE (8.2) will correspond to solutions of the original algebraic relations.

What we will attempt next is to try to reverse this procedure. That is to say, we will try to solve differential equations of the form (8.2) by first identifying the corresponding algebraic relation (8.1), and then solving (algebraically) for y.

The problem, however, is this: given a **general** non-linear, first order differential equation

$$M(x,y) + N(x,y)y' = 0 \tag{8.6}$$

how do we recognize that it's of the form (8.2), and more to the point, if it is of the form (8.2), how do we identify the function ψ that defines the corresponding algebraic relation?

It turns out that there is a rather simple criteria for seeing if a differential equation of the form (8.3) can be re-expressed in the form (8.2).

THEOREM 8.1. *Suppose that the functions M, N, $\frac{\partial M}{\partial y}$, and $\frac{\partial N}{\partial x}$ are all continuous in a region $R = \{(x,y) \mid \alpha < x < \beta \ , \gamma < y < \delta \ \}$. Then*

$$M(x,y) + N(x,y)y' = 0 \tag{8.7}$$

can be re-expressed in the form

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} y' = 0$$

at each point in R if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{8.8}$$

at each point in R.

In other words, there exists a function ψ such that

$$\begin{aligned} M &= \frac{\partial \psi}{\partial x} \\ N &= \frac{\partial \psi}{\partial y} \end{aligned} \tag{8.9}$$

if and only if (8.8) is satisfied. If (8.8) is satisfied then we say the differential equation is **exact**. The necessity of condition (8.8) is easy to see; for if M and N satisfy (8.9) then we have

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x}\frac{\partial \psi}{\partial y} = \frac{\partial N}{\partial x} \quad .$$

The sufficiency of condition (8.9) is a little more difficult to prove; but it amounts to a short calculation which more or less follows the pattern of the examples given below (see the textbook for more details).

EXAMPLE 8.2.

$$(2x+3) + (2y-2)y' = 0$$

We have

$$\begin{aligned} M(x,y) &= 2x+3 \\ N(x,y) &= 2y-2 \end{aligned}$$

and

$$\frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x}$$

so this equation is exact. We now try to find a function ψ such that its partial derivative with respect to x is M and its partial derivative with respect to y is N. Now the most general function ψ one can have such that

$$\frac{\partial \psi}{\partial x} = 2x+3$$

would be something of the form

$$\psi = x^2 + 3x + h_1(y). \tag{8.10}$$

(You should verify for yourself why this must be so.) Similarly, in order for

$$\frac{\partial \psi}{\partial y} = 2y-2$$

we must have

$$\psi = y^2 - 2y + h_2(x) \quad . \tag{8.11}$$

We now ask the question: is there any way we can make the forms (8.10) and (8.11) compatible with one another. Certainly, we simply set

$$\begin{aligned} h_1(y) &= y^2 \\ h_2(x) &= x^2 \end{aligned}$$

So we set

$$\psi = x^2 + 3x + y^2 - 2y \quad .$$

A general solution of the original differential equation is then constructed by solving

$$\psi = x^2 + 3x + y^2 - 2y = C$$

for y. One obtains

$$y = \frac{2 \pm \sqrt{4 - 4(x^2 + 3x - C)}}{2} \quad .$$

□

Example 8.3.

$$\frac{dy}{dx} = \left(\frac{ax - by}{bx - cy}\right)$$

This equation does not seem to be exact. However, if we re-write it in the form

$$-ax + by + (bx - cy)y' = 0$$

it is clearly exact since

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(-ax + by) = b = \frac{\partial}{\partial x}(bx - cy) = \frac{\partial N}{\partial x} \quad .$$

We therefore try to construct a function ψ such that

$$\frac{\partial \psi}{\partial x} = -ax + by \tag{8.12}$$

and

$$\frac{\partial \psi}{\partial y} = bx - cy \quad . \tag{8.13}$$

Now in order for (8.12) to hold, ψ must be of the form

$$\psi = -\frac{1}{2}ax^2 + bxy + h_1(y) \tag{8.14}$$

and for (8.13) to hold we must have

$$\psi = bxy - \frac{1}{2}cy^2 + h_2(x) \quad . \tag{8.15}$$

In order to satisfy both (8.14) and (8.15) we can take

$$\psi = -\frac{1}{2}ax^2 + bxy - \frac{1}{2}cy^2 \quad . \tag{8.16}$$

Setting

$$\psi = -\frac{1}{2}ax^2 + bxy - \frac{1}{2}cy^2 = k$$

and solving for y yields

$$y = \frac{bx \pm \sqrt{b^2x^2 - c\,(ax^2 + 2k)}}{c} \quad .$$

(The constant k corresponds to initial conditions: $k = \frac{cy_o^2}{2}$.)

□