LECTURE 7

Constants of Integration and Initial Conditions

0.1. Review. Recall from our last lecture that the general solution of a first order linear differential equation

$$y' + p(x)y = g(x) \tag{1}$$

has the form

$$y(x) = \frac{1}{\mu(x)} \int^x \mu(x')g(x')dx' + \frac{C}{\mu(x)} \tag{2a}$$

where

$$\mu(x) = \exp\left(\int^x p(x')dx'\right) \tag{2b}$$

and C is an arbitrary constant.

Remark 7.1. Since the variables over which we integrate drop out of the formulas once we evaluate the anti-derivatives of the integrand at the end points of integration, we often call them "dummy variables", as their actual labels are immaterial: We note, for example,

$$\begin{aligned} \int^x p(x')\,dx' &\equiv \text{ anti-derivatve of } p(x') \text{ evaluated at } x \\ &= \text{ anti-derivative of } p(s) \text{ evaluated at } x \\ &= \int^x p(s)\,ds \end{aligned}$$

The point of the writing, for example,

$$\int^x p(x')\,dx'$$

instead of

$$\int^x p(x)\,dx$$

is so that we can a clear distinction between the variable over which we are integrating and the variable that is to be used as an endpoint of integration.

There is another way of framing this result; one where the auxiliary function $\mu(x)$ is replaced with a more readily interpretable function. The reframing consists of two propositions.

Proposition 7.2. *The general solution of $y' + p(x)y = 0$ is given by*

$$y_0(x) = C\exp\left(-\int p(x)\,dx\right) \quad . \tag{3}$$

Proof. The differential equation is just a special case of equation (1) above (the case when $g(x) = 0$). The formula given for y_0 follows from (2a) and (2b) (with $g(x) = 0$):

$$y_o(x) = \frac{1}{\mu(x)} \int^x \mu(x') \cdot 0dx' + \frac{C}{\mu(x)} = 0 + \frac{C}{\exp(p(x)\,dx)} = C\exp\left(-\int p(x)\,dx\right)$$

Proposition 7.3. *The general solution of $y' + p(x)y = g(x)$ is given by*

$$y(x) = y_0(x) \int \frac{g(x)}{y_0(x)} dx + y_0(x) \tag{4}$$

where y_0 is any solution of $y' + p(x)y = 0$.

Proof. This is just a rewriting of the formulas (2a) and (2b) substituting

$$\frac{C}{y_0(x)} = \frac{C}{C\exp\left(-\int p(x)\,dx\right)} = \frac{1}{\exp\left(-\int p(x)\,dx\right)} = \exp\left(\int p(x)\,dx\right)$$

for $\mu(x)$.

In other words, in this new point of view, one solves

$$y' + p(x) = g(x)$$

by first finding the general solution y_0 of

$$y' + p(x)y = 0$$

(using the formula (3a)) and plugging this solution into the formula (4). Although this rephrasing may seem more complicated and unnecessary, this new way of thinking about first order linear differential equations will dovetail nicely into our study of second order linear differential equations.

0.2. First order linear ODEs and Initial Value Problems. In physical applications, one generally has the following situation. First, a differential equation that describes a particular class of phenomena (such as the trajectory of a ball thrown up in the air) is derived from general theoretical principles (e.g., Newton's 2^{nd} Law of Motion). Then particular instances of this class of phenomena are prescribed in terms of their initial conditions (e.g., the initial position and the initial velocity of the ball). Thus, in applications one tends to look for simultaneous solutions of a differential equation and a certain set of initial conditions. Such problems are referred to as **initial value problems**.

Example 7.4. Solve the following initial value problem:

$$\begin{aligned} y' + 2y &= xe^{-2x} \\ y(1) &= 0 \quad . \end{aligned}$$

Step 1. Solution of the differential equation.

Plugging into our general formula we quickly find

$$\begin{aligned} \mu(x) &= \exp\left[\int^x 2dx\right] = e^{2x} \\ y(x) &= \tfrac{1}{e^{2x}}\left[\int^x \left(e^{2x}\right)\left(xe^{-2x}\right)dx\right] + \tfrac{C}{e^{2x}} \\ &= \tfrac{x^2e^{-2x}}{2} + Ce^{-2x} \end{aligned}$$

Step 2. Plugging the result from Step 1 into the initial condition equation to fix the constant C.

$$\begin{aligned} 0 &= y(1) \\ &= \tfrac{1}{2} + C \\ \Rightarrow \quad C &= -\tfrac{1}{2} \\ \Rightarrow \quad y(x) &= \tfrac{x^2e^{-2x}}{2} - \tfrac{1}{2}e^{-2x} \\ &= \tfrac{1}{2}e^{-2x}\left(x^2 - 1\right) \quad . \end{aligned}$$

EXAMPLE 7.5. Find a solution to

$$y' + \frac{2}{x} y = \frac{\cos(x)}{x^2} \quad ,$$

on the interval $(0, +\infty)$ that satisfies the initial condition

$$y(\pi) = 0 \quad .$$

Since $p(x) = \frac{2}{x}$, one quickly calculates, exactly as in the example of the previous lecture, that

$$\mu(x) = \exp\left(\int^x \frac{2}{x'}\, dx'\right) = x^2 \quad ,$$

and so

$$\begin{aligned} y(x) &= \frac{1}{x^2}\int x^2\left(\frac{\cos(x)}{x^2}\right) dx + \frac{C}{x^2} \\ &= \frac{\sin(x)}{x^2} + \frac{C}{x^2} \quad . \end{aligned}$$

We will now fix the value of the constant C by imposing the initial condition $y(\pi) = 0$.

$$0 = y(\pi) = \frac{\sin(\pi) + C}{x^2} = \frac{C}{x^2} \quad \Rightarrow \quad C = 0 \quad .$$

So

$$y(x) = \frac{\sin(x)}{x^2} \quad .$$

There is an even more expedient way of solving the initial value problem; one in which are initial conditions are inserted directly into the formula for y(x).

Recall that the function $\mu(x)$ used in the formula for $y(x)$ was defined as the exponential of the anti-derivative of $p(x)$:

$$\mu(x) = \exp\left(\int^x p(x')\, dx'\right) \quad .$$

Since anti-derivatives are only determined up to the addition of an arbitrary constant, one is free to add to the term inside the parentheses an arbitrary number. And one particular way of doing this is by introducing a lower limit to the integration. Therefore, let us set

$$\mu_o(x) = \exp\left(\int_{x_o}^x p(x')\, dx\right) \quad .$$

Our claim is that if $\mu_o(x)$ is chosen in this way, then

$$y(x) = \frac{1}{\mu_o(x)}\int_{x_o}^x \mu_o(x') g(x')\, dx' \quad + \quad \frac{y_o}{\mu_o(x)}$$

is the unique solution of the initial value problem

$$\begin{aligned} y' + p(x)y &= g(x) \quad , \\ y(x_o) &= y_o \quad . \end{aligned}$$

We first show that

$$\lim_{x\to x_o} \mu_o(x) = 1 \quad .$$

This follows from the following observation:

$$\lim_{x\to x_o}\int_{x_0}^x p(x')\, dx' \le \lim_{x\to x_o}\left(\textit{area under curve } y = p(x) \textit{ between } x_o \textit{ and } x\right) = 0$$

so,

$$\lim_{x\to x_o} \mu_o(x) = \exp\left(\lim_{x\to x_o}\int_{x_0}^x p(x')\, dx'\right) = e^0 = 1 \quad .$$

Thus, when we write the general solution of (1) as

$$y(x) = \frac{1}{\mu_o(x)} \int_{x_o}^{x} \mu_o(x')g(x')\,dx + \frac{C}{\mu_o(x)} \tag{5}$$

and plug into the initial value equation

$$y(x_o) = y_o \quad ,$$

we get

$$y_o = \frac{1}{1} \cdot 0 + \frac{C}{1} = C \quad .$$

(Comparing (5) with (2), notice that we have also added a lower limit to the integral of $\mu(x)g(x)$. The $y(x)$ so defined is still a solution of the differential equation (1); the effect of adding the lower limit is equivalent to a harmless change in the arbitrary constant C.)

Thus, the unique solution to the initial value problem

$$\begin{aligned} y' + p(x)y &= g(x) \\ y(x_o) &= y_o \end{aligned}$$

is given by the formula

$$\begin{array}{c} y(x) = \frac{1}{\mu_o(x)} \int_{x_o}^{x} \mu_o(x')g(x')dx' + \frac{y_o}{\mu_o(x)} \\ where \qquad \\ \mu_o(x) = \exp\left(\int_{x_o}^{x} p(x')dx'\right) \qquad . \end{array} \tag{6}$$

EXAMPLE 7.6.

$$\begin{aligned} y' + \cot(x)y &= 3\csc(x) \qquad , \qquad \frac{\pi}{2} \leq x < \pi \quad . \\ y\left(\frac{\pi}{2}\right) &= 1 \end{aligned}$$

We first calculate $\mu_o(x)$.

$$\begin{aligned} \mu(x) &= \exp\left(\int_{\frac{\pi}{2}}^{x} \cot(x')\,dx'\right) \\ &= \exp\left(\ln\left(|\sin(x)| - \ln\left(|\sin(\frac{\pi}{2})|\right)\right)\right) \\ &= |\sin(x)| \quad . \\ &= \sin(x) \qquad \text{when } \frac{\pi}{2} \leq x < \pi \end{aligned}$$

We can now apply formula (6):

$$\begin{aligned} y(x) &= \frac{1}{\sin(x)} \int_{\frac{\pi}{2}}^{x} 2\sin(x')\csc(x')\,dx' + \frac{1}{\sin(x)} \\ &= \frac{1}{\sin(x)} \int_{\frac{\pi}{2}}^{x} 2\,dx' + \frac{1}{\sin(x)} \\ &= \frac{1}{\sin(x)} [2x - \pi + 1] \end{aligned}$$