LECTURE 3

Graphical Interpretation of First Order Differential Equations

Consider the graph of a solution $x(t)$ of the differential equation

$$\frac{dx}{dt} = F\left(x(t), t\right) \tag{3.1}$$

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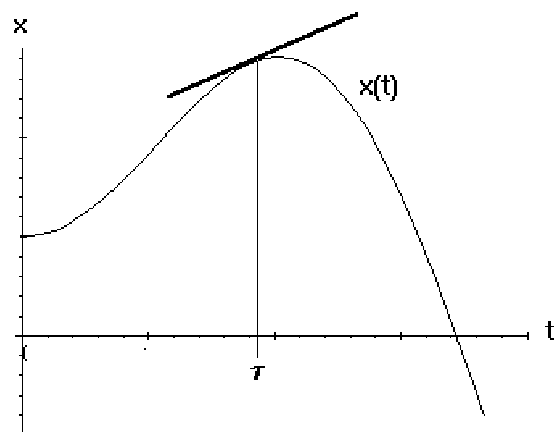
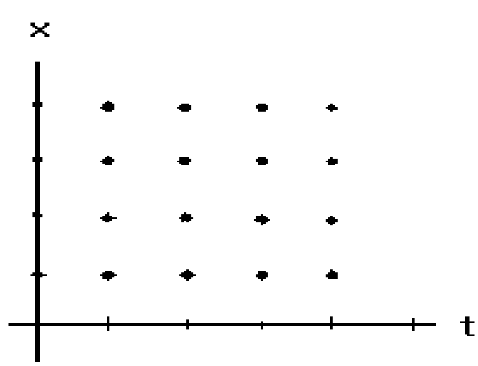
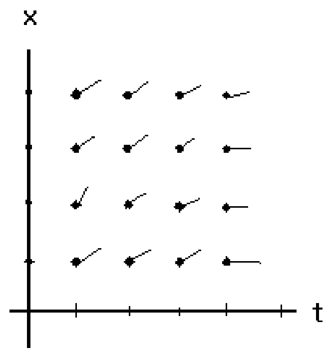


Now $\frac{dx}{dt}(\tau)$ is precisely the slope of the graph of $x(t)$ at the point $(\tau, x(\tau))$. Thus, since $x(t)$ is to be a solution of the differential equation (3.1), we can conclude the that slope of the graph of $x(t)$ at the point $(\tau, x(\tau))$ is exactly $F\left(x(\tau), \tau\right)$.

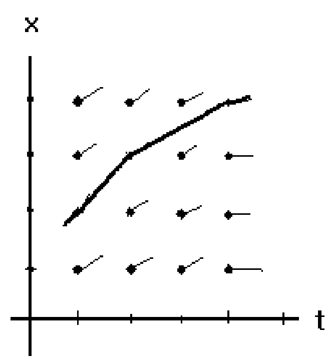
Now let's remove the graph of $x(t)$ from the picture, and look instead a grid of points in the tx-plane:



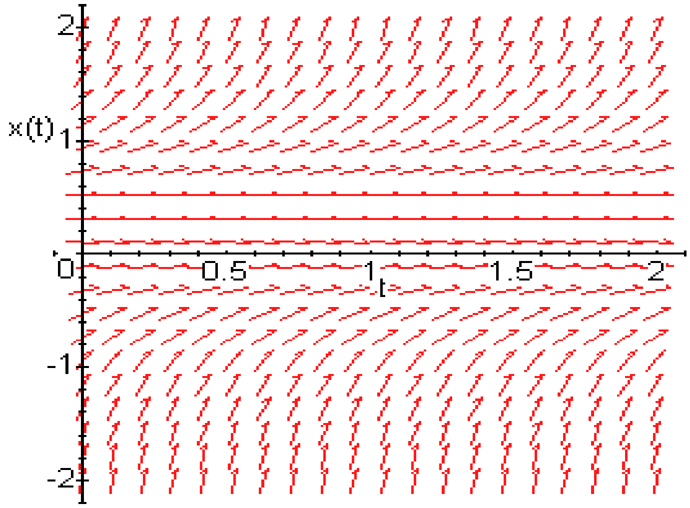
We still know that the slope of the solution that passes thru the point (t,x) must be given by $F(x,t)$.. Therefore, to get a picture of the possible solutions of the differential equation (3.1) we can pick a bunch of sample points (t_i,x_j) forming a nice rectangular grid in the tx-plane, calculate the value of $F(x,t)$ at each of these points, and then draw short lines with slopes $F(x_j,t_i)$ passing through the points $(t_i x_j)$



and then finally we can try to draw curves that pass thru all the points (t_i,x_j) in such a way that their tangent lines are always parallel to the lines eminating from each of the points (t_i,x_j).



If you do this for a large number of points you can get a fairly accurate picture of a large number of solutions of your differential equation.



The graph above corresponds to the differential equation

$$\frac{dx}{dt} = t\sin(x).$$

It was produced by Maple via the following commands:

(1) with(DEtools);
(2) dfieldplot(diff(x(t),t) = t*sin(x),[x],t=0..2,x=0..2);

0.1. Interpretation of Graphical Solutions. What's nice about the graphical method described above is that it gives a fairly accurate view of *all* solutions (in a given region of the tx-plane) of a first order differential equation. Of course accuracy here does not mean numerical accuracy. What I mean to say is that the picture itself is enough to provide accurate knowledge about the solutions.

Example 3.1. Sketch the direction fields associated with the following differential equation

$$\dot{x} = x(x-1)$$

Below is the output of the Maple command "dfieldplot(diff(x(t),t) = x*(2*x -1),[x],t=0..2,x=-2..2);":

Example 3.2. Now suppose this differential equation describes the position of a particle as a function of time. Can you make any predictions about the trajectories of particles as $t \to \infty$?

Let's look at the direction field plot. Note that at all points above the line $x = 1$, the direction field vectors have positive slope. This means the the solutions which have at least one point above the line $x = 1$ are always increasing (their tangent vectors always have positive slope). So any solution $x(t)$ that starts off above the line $x = 1$ will tend to infinity as t goes to infinity.

What about solutions that pass through the line $y = 1$? Well, the direction field vectors are identically zero along the line $x = 1$. So the slope of any solution $x(t)$ passing through the line $y = 1$ is constant and equal to zero. Therefore, once a solution reaches the line $x = 1$, it stays there.

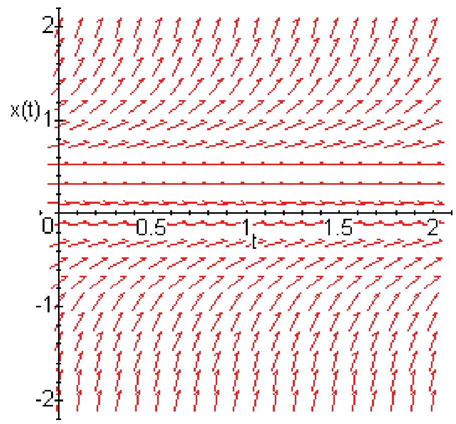


FIGURE 1

At this point, it might be helpful to look specifically at the sign of the function $F(x,t) = x(x-1)$ that defines the differential equation in the various regions of the xt-plane:

Region	$sign(\frac{dx}{dt}) = sign(F(x,t))$
$x > 1$	positive
$x = 1$	zero
$0 < x < 1$	negative
$x = 0$	zero
$x < 0$	positive

Thus, if a solution starts off in the region $x > 1$ then its slope is always positive, and so such a solution would tend to ∞ as $t \to \infty$.

If a solution starts off with $x = 1$, then its slope is initially zero, and so the function is initially constant. But then it can never leave the line $x = 1$. And so such a solution will just be the constant solution $x(t) = 1$

If a solution starts off with $0 < x < 1$, then its slope is initially negative, so the function is initially decreasing. However, at $x = 0$, the slope is zero again, so the solution cannot decrease any further. Such solutions will thus asymptotically approach the line $x = 0$ ast $t \to \infty$.

If a solution starts off with $x = 0$, then the slope is initially zero and remains at zero. Thus, such a solution will always be the constant solution $x(t) = 0$

If a solution starts off with $x < 0$, then its slope will be initially positive. However, such a solution can not increase past the value $x = 0$ since the slope must be zero along the line $x = 0$. Therefore, such a solution will asymptotically approach the line $x = 0$ as $t \to \infty$.