LECTURE 3

Graphical Interpretation of First Order Differential Equations

Consider the graph of a solution $x(t)$ of the differential equation

$$\frac{dx}{dt} = F(x(t), t)$$

Now $\frac{dx}{dt}(\tau)$ is precisely the slope of the graph of $x(t)$ at the point $(\tau, x(\tau))$. Thus, since $x(t)$ is to be a solution of the differential equation (3.1), we can conclude that the slope of the graph of $x(t)$ at the point $(\tau, x(\tau))$ is exactly $F(x(\tau), \tau)$.

Now let’s remove the graph of $x(t)$ from the picture, and look instead a grid of points in the $tx$-plane:
We still know that the slope of the solution that passes thru the point \((t, x)\) must be given by \(F(x, t)\). Therefore, to get a picture of the possible solutions of the differential equation (3.1) we can pick a bunch of sample points \((t_i, x_j)\) forming a nice rectangular grid in the \(tx\)-plane, calculate the value of \(F(x, t)\) at each of these points, and then draw short lines with slopes \(F(x_j, t_i)\) passing through the points \((t_i, x_j)\).

![Graphical interpretation of first order differential equations](image)

and then finally we can try to draw curves that pass thru all the points \((t_i, x_j)\) in such a way that their tangent lines are always parallel to the lines eminating from each of the points \((t_i, x_j)\).

![Graphical interpretation of first order differential equations](image)

If you do this for a large number of points you can get a fairly accurate picture of a large number of solutions of your differential equation.
The graph above corresponds to the differential equation

\[ \frac{dx}{dt} = t \sin(x). \]

It was produced by Maple via the following commands:

1. with(DEtools);
2. dfieldplot(diff(x(t),t) = t*sin(x),[x],t=0..2,x=0..2);

0.1. Interpretation of Graphical Solutions. What’s nice about the graphical method described above is that it gives a fairly accurate view of all solutions (in a given region of the tx-plane) of a first order differential equation. Of course accuracy here does not mean numerical accuracy. What I mean to say is that the picture itself is enough to provide accurate knowledge about the solutions.

Example 3.1. Sketch the direction fields associated with the following differential equation

\[ \dot{x} = x(x - 1) \]

Below is the output of the Maple command “dfieldplot(diff(x(t),t) = x*(2*x -1),[x],t=0..2,x=-2..2);”:

Example 3.2. Now suppose this differential equation describes the position of a particle as a function of time. Can you make any predictions about the trajectories of particles as \( t \to \infty \)?

Let’s look at the direction field plot. Note that at all points above the line \( x = 1 \), the direction field vectors have positive slope. This means the solutions which have at least one point above the line \( x = 1 \) are always increasing (their tangent vectors always have positive slope). So any solution \( x(t) \) that starts off above the line \( x = 1 \) will tend to infinity as \( t \) goes to infinity.

What about solutions that pass through the line \( y = 1 \)? Well, the direction field vectors are identically zero along the line \( x = 1 \). So the slope of any solution \( x(t) \) passing through the line \( y = 1 \) is constant and equal to zero. Therefore, once a solution reaches the line \( x = 1 \), it stays there.
At this point, it might be helpful to look specifically at the sign of the function $F(x,t) = x(x-1)$ that defines the differential equation in the various regions of the $xt$-plane:

<table>
<thead>
<tr>
<th>Region</th>
<th>$\text{sign} \left( \frac{dx}{dt} \right)$</th>
<th>$\text{sign} (F(x,t))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x &gt; 1$</td>
<td>positive</td>
<td>positive</td>
</tr>
<tr>
<td>$x = 1$</td>
<td>zero</td>
<td>zero</td>
</tr>
<tr>
<td>$0 &lt; x &lt; 1$</td>
<td>negative</td>
<td>negative</td>
</tr>
<tr>
<td>$x = 0$</td>
<td>zero</td>
<td>zero</td>
</tr>
<tr>
<td>$x &lt; 0$</td>
<td>positive</td>
<td>positive</td>
</tr>
</tbody>
</table>

Thus, if a solution starts off in the region $x > 1$ then its slope is always positive, and so such a solution would tend to $\infty$ as $t \to \infty$.

If a solution starts off with $x = 1$, then its slope is initially zero, and so the function is initially constant. But then it can never leave the line $x = 1$. And so such a solution will just be the constant solution $x(t) = 1$

If a solution starts off with $0 < x < 1$, then its slope is initially negative, so the function is initially decreasing. However, at $x = 0$, the slope is zero again, so the solution cannot decrease any further. Such solutions will thus asymptotically approach the line $x = 0$ ast $t \to \infty$.

If a solution starts off with $x = 0$, then the slope is initially zero and remains at zero. Thus, such a solution will always be the constant solution $x(t) = 0$

If a solution starts off with $x < 0$, then its slope will be initially positive. However, such a solution can not increase past the value $x = 0$ since the slope must be zero along the line $x = 0$. Therefore, such a solution will asymptotically approach the line $x = 0$ as $t \to \infty$. 