Math 2233 SOLUTION TO FINAL EXAM
10:00 – 11:50 am , May 4, 2011

1. (10 pts) Use the Euler (numerical) method with $\Delta x = 0.1$ to estimate $y(1.2)$, where $y(x)$ is the solution of

$$y' = x(1+y) \qquad , \qquad y(1) = 0 \quad .$$

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$$\begin{aligned}
x_0 &= 1 \\
y_0 &= 0 \\
x_1 &= x_0 + \Delta x = 1.1 \\
y_1 &= y_0 + x_0\,(1+y_0)\,\Delta x = 0 + (1)\,(1+0)\,(0.1) = 0.1 \\
x_2 &= x_1 + \Delta x = 1.2 \\
y_2 &= y_1 + x_1\,(1+y_1)\,\Delta x = 0.1 + (1.1)\,(1+0.1)\,(0.1) = 0.121
\end{aligned}$$

$$y\,(1.2) \approx 0.221$$

2. (15 pts) Find general solution of $xy' - 2y = x$.

- This is a first order linear equation equivalent to the following ODE in standard form

$$y' - \frac{2}{x}y = 1 \quad \Rightarrow \quad p\,(x) = -\frac{2}{x} \quad , \quad g\,(x) = 1$$

We thus have

$$\mu\,(x) = \exp\left(\int p\,(x)\,dx\right) = \exp\left(\int\left(-\frac{2}{x}\right)dx\right) = \exp\left(-2\ln|x|\right) = x^{-2}$$

and so

$$\begin{aligned}
y\,(x) &= \frac{1}{\mu\,(x)}\int \mu\,(x)\,g\,(x)\,dx + \frac{C}{\mu\,(x)} = x^2\int x^{-2}\,(1)\,dx + Cx^2 \\
&= x^2\left(-\frac{1}{x}\right) + Cx^2 \\
&= -x + Cx^2
\end{aligned}$$

- 3. (15 pts) Show that the following equation is exact and find an implicit solution.

$$2x + \frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = 0$$

We have

$$\begin{aligned}
M &= 2x + \frac{1}{x} \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 0 \\
N &= \frac{1}{y} \qquad \Rightarrow \quad \frac{\partial N}{\partial x} = 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ the equation is exact (in fact, it's separable). This means that the solutions of the differential equations are also solutions of algebraic equations of the form $\Phi\,(x,y) = C$, with $\Phi\,(x,y)$ a function of x and y such that

$$\begin{aligned}
\Phi\,(x,y) &= \int M\,(x,y)\,\partial x + h_1\,(y) = \int\left(2x + \frac{1}{x}\right)\partial x + h_1\,(y) = x^2 + \ln|x| + h_1\,(y) \\
\Phi\,(x,y) &= \int N\,(x,y)\,\partial y + h_2\,(x) = \int \frac{1}{y}\partial y + h_2\,(x) = \ln|y| + h_2\,(x)
\end{aligned}$$

Comparing these two expressions for Φ we see that we should take $h_1\,(y) = 0$ and $h_2\,(x) = x^2$ so that they agree. Thus, the implicit solution of the differential equation will be given by

$$x^2 + \ln|x| + \ln|y| = C$$

4. (15 pts) Use the Method of Variation of Parameters to find the general solution of the following inhomogeneous differential equation.

$$y'' - 3y' + 2y = e^{-x} \quad .$$

- First we solve the corresponding homogeneous equation:

$$y'' - 3y' + 2y = 0$$

This is 2^{nd} order linear with constant coefficients. Its characteristic equation is

$$0 = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2) \quad \Rightarrow \quad \lambda = 1, 2$$

So

$$y_1(x) = e^x \quad , \quad y_2(x) = e^{2x}$$

will be two independent solutions of $y'' - 3y' + 2y = 0$. We have

$$W[y_1, y_2] = (e^x)(2e^{ex}) - (e^x)\left(e^{2x}\right) = e^{3x}$$

and so

$$\begin{aligned} y_p(x) &= -y_1 \int \frac{y_2(x)\, g(x)}{W[y_1, y_2]} dx + y_2 \int \frac{y_1(x)\, g(x)}{W[y_1, y_2]} dx \\ &= -e^x \int \frac{e^{2x}\left(e^{-x}\right)}{e^{3x}} dx + e^{2x} \int \frac{e^x\left(e^{-x}\right)}{e^{3x}} dx \\ &= -e^{-x} \int e^{-2x} dx + e^{2x} \int e^{-3x} dx \\ &= -e^{-x}\left(-\frac{1}{2}e^{-2x}\right) + e^{2x}\left(-\frac{1}{3}e^{-3x}\right) \\ &= \frac{1}{6}e^{-x} \end{aligned}$$

The general solution will thus be

$$y(x) = \frac{1}{6}e^{-x} + c_1 e^x + c_2 e^{2x}$$

5. (10 pts) Find the general solution of the following differential equation.

$$y'''' - 4y'' = 0 \quad .$$

- This is a 4^{th} order linear differential equation with constant coefficients. Its characteristic equation is

$$0 = \lambda^4 - 4\lambda^2 = \left(\lambda^2\right)\left(\lambda^2 - 4\right) = (\lambda - 0)^2 (\lambda - 2)(\lambda + 2)$$

which has three distinct roots: $\lambda = 0$ with multiplicity 2, $\lambda = 2$ with multiplity 1, and $\lambda = -2$ with multiplicity 1. The general solution of the differential equation is thus

$$y(x) = c_1 e^{0x} + c_2 x e^{0x} + c_3 e^{2x} + c_4 e^{-2x} = c_1 + c_2 x + c_3 e^{2x} + c_4 e^{-2x} \quad .$$

6. (10 pts) Reduce the following expression to a single power series expression.

$$x^2 \sum_{n=0}^{\infty} n(n-1) a_n (x-1)^{n-2} + \sum_{n=0}^{\infty} a_n (x-1)^n$$

- First we compute the Taylor expansion of $f(x) = x^2$ about $x_0 = 1$. We have

$$f(1) = \left. x^2 \right|_{x=1} = 1 \quad , \quad f'(1) = \left. 2x \right|_{x=1} = 2 \quad , \quad f''(1) = \left. 2 \right|_{x=1} = 2$$

with all higher derivatives vanishing. Thus

$$x^2 = f(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}\left(x - 1^2\right) = 1 + 2(x-1) + (x-1)^2 \quad .$$

Consider now the first sum

$$x^2 \sum_{n=0}^{\infty} n(n-1) a_n (x-1)^{n-2} = \left[1 + 2(x-1) + (x-1)^2\right] n(n-1) a_n (x-1)^{n-2}$$

$$= \sum_{n=0}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=0}^{\infty} 2n(n-1)a_n(x-1)^{n-1} + \sum_{n=0}^{\infty} n(n-1)a_n(x-1)^n$$

$$= \sum_{n=-2}^{\infty} (n+2)(n+1)a_{n+2}(x-1)^n + \sum_{n=-1}^{\infty} 2(n+1)(n)a_{n+1}(x-1)^{n-1} + \sum_{n=0}^{\infty} n(n-1)a_n(x-1)^n$$

$$= 0 + 0 + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-1)^n + 0 + \sum_{n=0}^{\infty} 2(n+1)(n)a_{n+1}(x-1)^{n-1} + \sum_{n=0}^{\infty} 2n(n-1)a_n(x-1)^n$$

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + 2n(n+1)a_{n+1} + n(n-1)a_n\right](x-1)^n$$

The last expression on the right can now be readily added to the second sum in the original expression to yield

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + 2n(n+1)a_{n+1} + n(n-1)a_n + a_n\right](x-1)^n$$

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7.(15 pts) Find the recursion relations for a power series solution about $x_o = 1$ for the following differential equation.

$$xy'' - 2y = 0$$

- We look for solutions of the form $y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$. We have

$$\begin{aligned}
xy'' &= x \sum_{n=0}^{\infty} n(n-1) a_n (x-1)^{n-2} = [1 + (x-1)] \sum_{n=0}^{\infty} n(n-1) a_n (x-1)^{n-2} \\
&= \sum_{n=0}^{\infty} n(n-a) a_n (x-1)^{n-2} + \sum_{n=0}^{\infty} n(n-1) a_n (x-1)^{n-1} \\
&= 0 + 0 + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n + 0 + \sum_{n=0}^{\infty} (n+1)(n) a_{n+1} (x-1)^n
\end{aligned}$$

So when we plug the power series expression $\sum_{n=0}^{\infty} a_n (x-1)^n$ into the differential equation we get

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n + \sum_{n=0}^{\infty} (n+1)(n) a_{n+1} (x-1)^n + \sum_{n=0}^{\infty} (-2) a_n (x-1)^n \\
&= \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + n(n+1) a_{n+1} - 2a_n] (x-1)^n
\end{aligned}$$

This last equation requires

$$(n+2)(n+1) a_{n+2} + n(n+1) a_{n+1} - 2a_n = 0 \quad , \quad n = 0, 1, 2, 3, \dots$$

or

$$a_{n+2} = \frac{2a_n - n(n+1) a_n}{(n+2)(n+1)} \quad , \quad n = 0, 1, 2, 3, \dots$$

8. (10 pts) **Given** that the recursion relations for $y'' - xy' + y = 0$ about $x_o = 0$ are

$$a_{n+2} = \frac{(n-1)a_n}{(n+2)(n+1)} \quad , \quad n = 0, 1, 2, 3, \dots$$

Write down the first 4 terms of the power series solution satisfying $y(0) = 1$, $y'(0) = 2$ (i.e., find the solution up to order x^3.)

- If $y(x) = \sum_{n=0}^{\infty} a_n x^n$, the initial conditions imply

$$\begin{aligned}
a_0 &= y(0) = 1 \\
a_1 &= y'(0) = 2
\end{aligned}$$

The rest of the coefficients $a_2, a_3, a_4, \dots$ can now be determined by applying the recursion relations:

$$\begin{aligned}
a_2 = a_{0+2} &= \frac{(0-1) a_0}{(0+2)(0+1)} = \frac{(-1)(1)}{(2)(1)} = -\frac{1}{2} \\
a_3 = a_{1+2} &= \frac{(1-1) a_1}{(1+2)(1+1)} = 0
\end{aligned}$$

That's all the coefficients we need to get the solution up to order x^3. We thus obtain

$$\begin{aligned}
y(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\
&= 1 + 2x - \frac{1}{2} x^2 + \mathcal{O}(x^4)
\end{aligned}$$

9. Consider the following differential equation:

$$x^2 y'' + \frac{x}{(x+1)(x+2)^2} y' + \frac{1}{x(x+1)} y = 0$$

(a) (10 pts) Identify and classify (as *regular* or *irregular*) the singular points of this differential equation.

- For this differential equation we have

$$p(x) = \frac{1}{x(x+1)(x+2)^2} \quad , \quad q(x) = \frac{1}{x^3(x+1)}$$

Looking at the denominators of $p(x)$ and $q(x)$ we see that we have singular points at $x = 0$, $x = -1$ and $x = -2$.

x_x	$\deg(p(x), x_s)$	$\deg(q(x), x_s)$	$\deg(p(x), x_s) \leq 1$	$\deg(q(x)) \leq 2$	*type*
0	1	3	*true*	*false*	irregular
-1	1	1	*true*	*true*	regular
-2	2	0	*false*	*true*	irregular

(b) (10 pts) For what range of x is a power series solution about $x_o = -4$ guaranteed to converge? (Hint: what is the radius of convergence of a power series solution about $x_o = -4$?)

- The singular point that is closest to the expansion point is $x_s = -2$. Since the distance between $x_s = -2$ and $x_0 = -4$ is 2, we can conclude that the radius convergence of a power series solution will be (at least) 2 and that solution will be valid for all x in the range

$$x \in (x_0 - R, x_0 + R) = (-4 - 2, -4 + 2) = (-6, -2)$$

10. (10 pts) Find the function $f(x)$ whose Laplace transform is $\mathcal{L}[f](s) = \dfrac{2s+1}{s^2+2s+3}$. (Hint: use the Laplace transform tables on the last page.)

- $$\mathcal{L}[f](s) = \frac{2s+1}{s^2+2s+3} = \frac{2s+1}{s^2+2s+1+2} = \frac{2s+1}{(s+1)^2+\left(\sqrt{2}\right)^2}$$

  The denominator on the far right looks like that of either $\mathcal{L}\left[e^{-x}\sin\left(\sqrt{2}x\right)\right]$ or $\mathcal{L}\left[e^{-x}\cos\left(\sqrt{2}x\right)\right]$. We need to do a little more work to get the numerators to match up.

  $$\begin{aligned}\frac{2s+1}{(s+1)^2+\left(\sqrt{2}\right)^2} &= \frac{2s+2-1}{(s+1)^2+\left(\sqrt{2}\right)^2} = \frac{2s+2}{(s+1)^2+\left(\sqrt{2}\right)^2} - \frac{1}{(s+1)^2+\left(\sqrt{2}\right)^2} \\ &= 2\frac{s+1}{(s+1)^2+\left(\sqrt{2}\right)^2} - \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{(s+1)^2+\left(\sqrt{2}\right)^2} \\ &= 2\mathcal{L}\left[e^{-x}\cos\left(\sqrt{2}x\right)\right] - \frac{1}{\sqrt{2}}\mathcal{L}\left[e^{-x}\sin\left(\sqrt{2}x\right)\right] \\ &= \mathcal{L}\left[2e^{-x}\cos\left(\sqrt{2}x\right) - \frac{1}{\sqrt{2}}e^{-x}\sin\left(\sqrt{2}x\right)\right]\end{aligned}$$

  So

  $$f(x) = 2e^{-x}\cos\left(\sqrt{2}x\right) - \frac{1}{\sqrt{2}}e^{-x}\sin\left(\sqrt{2}x\right)$$

11. (15 pts) Use the Laplace transform method to solve the following differential equation with initial conditions

$$\begin{aligned} y'' - 5y' + 6 &= 0 \\ y(0) &= 1 \\ y'(0) &= 1 \end{aligned}$$

- Taking the Laplace transform of the differential equation yields

  $$s^2\mathcal{L}[y] - sy(0) - y'(0) - 5(s\mathcal{L}[y] - y(0)) + 6\mathcal{L}[y] = 0$$

  Applying the initial conditions and collecting terms proportional to $\mathcal{L}[y]$ we get

  $$\left(s^2 - 5 + 6\right)\mathcal{L}[y] - s - 1 + 5 = 0$$

  or

  $$\mathcal{L}[y] = \frac{s-4}{s^2-5+6} = \frac{s-4}{(s-2)(s-3)}$$

  We now have to figure out what function has the right hand side as its Laplace transform. Since the denominator factorizes we'll rewrite the right hand side in terms of a partial fractions expansion.

  $$\frac{s-4}{(s-2)(s-3)} = \frac{A}{s-2} + \frac{B}{s-3} \quad \Rightarrow \quad s-4 = A(s-3) + B(s-2)$$

  We now use a couple convenient values of s to figure out the constants A and B.

  $$\begin{aligned} s=2 &\quad\Rightarrow\quad 2-4 = A(2-3) + B(2-2) \quad\Rightarrow\quad -2 = -A \quad\Rightarrow\quad A = 2 \\ s=3 &\quad\Rightarrow\quad 3-4 = A(3-3) + B(3-2) \quad\Rightarrow\quad -1 = B \quad\Rightarrow\quad B = -1 \end{aligned}$$

  Thus we have

  $$\mathcal{L}[y] = \frac{s-4}{(s-2)(s-3)} = \frac{2}{s-2} - \frac{1}{s-3} = 2\mathcal{L}\left[e^{2x}\right] - \mathcal{L}\left[e^{3x}\right] = \mathcal{L}\left[2e^{2x} - e^{3x}\right]$$

  and so we can conclude that the solution of the original differential equation / boundary value problem is

  $$y(x) = 2e^{2x} - e^{3x}$$

Table of Laplace Transforms

$$\mathcal{L}\left[x^n\right] = \frac{n!}{s^{n+1}}$$
$$\mathcal{L}\left[e^{ax}\right] = \frac{1}{s-a}$$
$$\mathcal{L}\left[\sin(ax)\right] = \frac{a}{s^2+a^2}$$
$$\mathcal{L}\left[\cos(ax)\right] = \frac{s}{s^2+a^2}$$
$$\mathcal{L}\left[\sinh\left(ax\right)\right] = \frac{a}{s^2-a^2}$$
$$\mathcal{L}\left[\cosh\left(ax\right)\right] = \frac{s}{s^2-a^2}$$
$$\mathcal{L}\left[x^n e^{ax}\right] = \frac{n!}{\left(s-a\right)^{n+1}}$$
$$\mathcal{L}\left[e^{ax}\sin\left(bx\right)\right] = \frac{b}{\left(s-a\right)^2+b^2}$$
$$\mathcal{L}\left[e^{ax}\cos\left(bx\right)\right] = \frac{s-a}{\left(s-a\right)^2+b^2}$$
$$\mathcal{L}\left[e^{ax}\sinh\left(bx\right)\right] = \frac{b}{\left(s-a\right)^2-b^2}$$
$$\mathcal{L}\left[e^{ax}\cosh\left(bx\right)\right] = \frac{s-a}{\left(s-a\right)^2-b^2}$$