Math 2233
SOLUTIONS TO FINAL EXAM
10:00-11:50, Friday, May 6, 2005

1. (15 pts) Solve the following initial value problem.

$$xy' - 3y = x^2 \qquad , \qquad y(1) = 3 \quad .$$

- This is a first order linear equation with $p(x) = -3/x$ and $g(x) = x$.

$$\begin{aligned} \mu(x) &= \exp\left[\int p(x)dx\right] = \exp\left[\int -\frac{3}{x}dx\right] = \exp\left(-3\ln|x|\right) = x^{-3} \\ y(x) &= \frac{1}{\mu}\int \mu g dx + \frac{C}{\mu} = x^3\int \left(x^{-3}\right)(x)\,dx + Cx^3 = x^3\left(-x^{-1}\right) + Cx^3 = -x^2 + Cx^3 \end{aligned}$$

Initial conditions require

$$\begin{aligned} 3 &= y(1) = -(1)^2 + C\,(1)^3 = -1 + C \qquad \Rightarrow \qquad C = 4 \\ &\Rightarrow \qquad y = -x^2 + 4x^3 \end{aligned}$$

□

2. Consider the following differential equation.

$$\left(y + 6x^2\right)dx + \left(x\ln(x) - 4xe^{2y}\right)dy = 0.$$

(a) (5 pts) Verify that $\mu = 1/x$ is an integrating factor for this equation.

- Multiplying the equation by $1/x$ we obtain

$$\begin{aligned} &\Rightarrow \qquad \left(\frac{y}{x} + 6x\right)dx + \left(\ln(x) - 4e^{2y}\right)dy = 0 \\ &\Rightarrow \qquad M(x,y) = \frac{y}{x} + 6x \quad , \quad N(x,y) = \ln(x) - 4e^{2y} \\ &\Rightarrow \qquad \frac{\partial M}{\partial y} = \frac{1}{x} = \frac{\partial N}{\partial x} \qquad \Rightarrow \qquad \text{the equation is exact} \end{aligned}$$

□

(b) (10 pts) Find an implicit solution for this differential equation.

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$$\left.\begin{aligned} \Phi = \int M dx + H_1(y) \qquad &\Rightarrow \qquad \Phi = y\ln|x| + 3x^2 + H_1(y) \\ \Phi = \int N dy + H_2(x) \qquad &\Rightarrow \qquad \Phi = y\ln|x| - 2e^{2y} + H_2(x) \end{aligned}\right\} \qquad \Rightarrow \qquad \Phi = y\ln|x| + 3x^3 - 2e^{2y}$$

$$\Rightarrow \qquad y\ln|x| + 3x^3 - 2e^{2y} = C$$

□

3. (15 pts) Find an explicit solution of the following homogeneous ODE: $\dfrac{dy}{dx} = \dfrac{y^2 + yx}{x^2}$.
(Hint: use the following change of variables: $z = y/x$.)

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$$z = \frac{y}{x} \quad \Rightarrow \quad \begin{cases} y = zx \\ y' = z'x + z \end{cases}$$

$$\Rightarrow \quad z'x + z = \frac{(zx)^2 + (zx)x}{x^2} = z^2 + z$$

$$\Rightarrow \quad z'x = z^2 \quad \Rightarrow \quad \frac{dz}{z^2} = \frac{dx}{x} \quad \Rightarrow \quad -z^{-1} = \ln|x| + C$$

$$\Rightarrow \quad -\frac{x}{y} = \ln|x| + C \quad \Rightarrow \quad y = \frac{-x}{\ln|x| + C}$$

□

4. Find the general solutions of the following differential equations.
(a) (5 pts) $y'' - 3y' + 3y = 0$
– This is second order linear with constant coefficients. Substituting $y = e^{\lambda x}$ we find

$$0 = \lambda^2 - 3\lambda + 3 \quad \Rightarrow \quad \lambda = \frac{3 \pm \sqrt{9-12}}{2} = \frac{3}{2} \pm \frac{\sqrt{3}}{2} i$$

$$\Rightarrow \quad y = c_1 e^{\frac{3}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 c_1 e^{\frac{3}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right)$$

□

(b) (5 pts) $x^2 y'' - 5xy' + 9y = 0$
– This is a Euler-type equation. Substituting $y = x^r$ we find

$$0 = r(r-1) - 6r + 9 = r^2 - 6r + 9 = (r-3)^2 \quad \Rightarrow \quad r = 3$$

$$\Rightarrow \quad y = c_1 x^3 + c_2 x^3 \ln|x|$$

□

(c) (5 pts) $y'''' + 4y''' + 4y'' = 0$.
– This is fourth order linear with constant coefficients. Substituting $y = e^{\lambda x}$ we find

$$0 = \lambda^4 + 4\lambda^3 + 4\lambda^2 = \lambda^2\left(\lambda^2 + 4\lambda + 4\right) = \lambda^2 (\lambda + 2)^2 \quad \Rightarrow \quad \lambda = 0, -2$$

$$\Rightarrow \quad y = c_1 + c_2 x + c_3 e^{-2x} + c_4 x e^{-2x}$$

□

5. (10 pts) Given that $y_1(x) = x$ is one solution of $x^2y'' - xy' + y = 0$, use Reduction of Order to determine the general solution.

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$$\begin{aligned} y_2 &= y_1 \int \frac{1}{[y_1]^2} \exp\left(-\int^x pds\right) dx = x \int x^{-2} \exp\left(\int^x \frac{1}{s} ds\right) dx = x \int x^{-2} \exp\left[\ln|x|\right] dx = x \int x^{-2} (x)\, dx = x \int x^{-1} \\ &= x \ln|x| \end{aligned}$$

$$\Rightarrow \qquad y = c_1 x + c_2 x \ln|x|$$

□

6. (15 pts) Use the Method of Variation of Parameters to find the general solution of the following inhomogeneous differential equation.

$$y'' - 5y' + 6y = e^x \quad .$$

– The homogeneous equation is

$$y'' - 4y' + 6y = 0$$

This is 2^{nd} order with constant coefficients. Substituting $y = e^{\lambda x}$ into this ODE we get

$$\begin{aligned} 0 &= \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) \qquad \Rightarrow \qquad \lambda = 2, 3 \\ &\Rightarrow \qquad y_1 = e^{3x} \quad , \quad y_2 = e^{2x} \end{aligned}$$

$$\begin{aligned} W[y_1, y_2] &= y_1 y_2' - y_1' y_2 = \left(e^{2x}\right)\left(3e^{3x}\right) - \left(2e^{2x}\right)\left(e^{3x}\right) \\ &= e^{5x} \end{aligned}$$

$$\begin{aligned} y_p &= -y_1 \int \frac{y_2 g}{W[y_1, y_2]} dx + y_2 \int \frac{y_1 g}{W[y_1, y_2]} dx \\ &= -e^{2x} \int \frac{e^{3x} e^x}{e^{5x}} dx + e^{3x} \int \frac{e^{2x} e^x}{e^{5x}} dx \\ &= -e^{2x} \int e^{-x} dx + e^{3x} \int e^{-2x} dx \\ &= -e^{2x} \left(\frac{1}{-1} e^{-x}\right) + e^{3x} \left(\frac{1}{-2} e^{-2x}\right) = e^x \left(1 - \frac{1}{2}\right) = \frac{1}{2} e^x \end{aligned}$$

$$y(x) = y_p(x) + c_1 y_1(x) + c_2 y_2(x) = \frac{1}{2} e^x + c_1 e^{2x} + c_2 e^{3x}$$

□

7. (10 pts) Suppose $y = \sum_{n=0}^{\infty} a_n(x-1)^n$ is a series solution of $xy'' + 2y = 0$. Determine the recursion relations for the coefficients $\{a_n\}$.

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$$\begin{aligned}
2y &= \sum_{n=0}^{\infty} 2a_n (x-1)^n \\
xy'' &= ((x-1)+1)\sum_{n=0}^{\infty} n(n-1)a_n(x-1)^{n-2} = \sum_{n=0}^{\infty} n(n-1)a_n(x-1)^{n-1} + \sum_{n=0}^{\infty} n(n-1)a_n(x-1)^{n-2} \\
&= \sum_{n=-1}^{\infty} (n+1)(n)a_{n+1}(x-1)^n + \sum_{n=-2}^{\infty} (n+2)(n+1)a_{n+1}(x-1)^n \\
&= 0 + \sum_{n=0}^{\infty} (n+1)(n)a_{n+1}(x-1)^n + 0 + 0 + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+1}(x-1)^n \\
&= \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + (n+1)na_{n+1}](x-1)^n \\
0 &= xy'' - 2y \qquad \Rightarrow \qquad 0 = \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + (n+1)na_{n+1} - 2a_n](x-1)^n
\end{aligned}$$

$$\Rightarrow \qquad 0 = (n+2)(n+1)a_{n+2} + (n+1)na_{n+1} - 2a_n \qquad , \qquad n = 0, 1, 2, 3, \ldots$$

$$\Rightarrow \qquad a_{n+2} = \frac{2a_n - n(n+1)a_{n+1}}{(n+2)(n+1)}$$

□

8. (10 pts) Suppose that $y(x) = \sum_{n=0}^{\infty} a_n x^n$ is a power series solution of $y'' - xy' - y = 0$. Given that the recursion relations for the coefficients $\{a_n\}$ are

$$a_{n+2} = \frac{a_n}{n+2} \quad ,$$

write down the first four terms of the power series solution satisfying $y(0) = 3$, $y'(0) = 2$.

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$$\begin{aligned}
a_0 &= 3 \\
a_1 &= 2 \\
a_2 &= \frac{a_0}{0+2} = \frac{3}{2} \\
a_3 &= \frac{a_1}{1+2} = \frac{2}{3}
\end{aligned}$$

$$\Rightarrow \qquad y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \quad = 3 + 2x + \frac{3}{2}x^2 + \frac{2}{3}x^3 + \cdots$$

□

9. Consider the differential equation $x^2(x+2)^2 y'' - 2xy' + 3y = 0$

(a) (10 pts) Identify and classify the singular points of this differential equation.

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$$\left.\begin{aligned} p(x) &= \tfrac{-2}{x(x+2)^2} \\ q(x) &= \tfrac{3}{x^2(x+2)^2} \end{aligned}\right\} \qquad \Rightarrow \qquad \text{singular points are } x = 0, -2$$

$$\begin{aligned}
\deg(p,0) &= 1 \le 1 \quad , \deg(q,0) = 2 \le 2 \qquad \Rightarrow \qquad x = 0 \text{ is a regular singular point} \\
\deg(p,-2) &= 2 > 1 \quad , \deg(q,-2) = 2 \qquad \Rightarrow \qquad x = 0 \text{ is a irregular singular point}
\end{aligned}$$

□

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(b) (5 pts) What is the minimal radius of convergence of a power series solution of the form $y(x) = \sum_{n=0}^{\infty} a_n(x-4)^n$?

- The expansion point is $x_0 = 4$ and the singular point that's closest to that point is $x = 0$
$$R_{\min} = \min \{\|4 - 0\|, \|4 - (-2)\|\} = \min \{4, 6\} = 4$$

$\square$

10. (15 pts) Consider the differential equation

$$3xy'' + y = 0$$

and suppose there is a series solution of the form

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n \quad \text{with } a_0 \neq 0 \quad .$$

Determine the leading exponents (the possible values for r) and the recursion relations for the coefficients a_n. (You **do not** have to write down the corresponding solutions.)

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$$\begin{aligned}
y &= x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} \\
3xy'' &= 3x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} = \sum_{n=0}^{\infty} 3(n+r)(n+r-1) a_n x^{n+r-1} = \sum_{n=-1}^{\infty} 3(n+r+1)(n+r) a_{n+1} x^{n+r} \\
&= 3r(r-1)a_0 + \sum_{n=0}^{\infty} 3(n+r+1)(n+r) a_{n+1} x^{n+r}
\end{aligned}$$

$$\begin{aligned}
0 &= 3xy'' + y = 3r(r-1) + \sum_{n=0}^{\infty} \left[3(n+r+1)(n+r) a_{n+1} + a_n\right] x^{n+r} \\
&\Rightarrow \quad \begin{cases} 0 = r(r-1) \\ 0 = 3(n+r+1)(n+r) a_{n+1} + a_n \quad , \quad n = 0, 1, 2, \dots \end{cases}
\end{aligned} \tag{1}$$

Thus the possible values of r are 0 and 1, and the recursion relations are

$$a_{n+1} = \frac{-a_n}{3(n+r+1)(n+r)}$$

□

11. (15 pts) Use the Laplace Transform Method to solve

$$\begin{aligned}
y'' + 3y' + 2y &= 0 \\
y(0) &= 3 \\
y'(0) &= -4
\end{aligned}$$

– Taking the Laplace transform of the differential equation we obtain

$$\begin{aligned}
0 &= \left(s^2\mathcal{L}[y] - sy(0) - y'(0)\right) + 3\left(s\mathcal{L}[y] - y(0)\right) + 2\mathcal{L}[y] = s^2\mathcal{L}[y] - 3s + 4 + 3s\mathcal{L}[y] - 9 + 2\mathcal{L}[y] \\
&= \left(s^2 + 3s + 2\right)\mathcal{L}[y] - 3s - 5 \\
\Rightarrow & \qquad \mathcal{L}[y] = \frac{3s+5}{s^2+3s+2} = \frac{3s+5}{(s+1)(s+2)}
\end{aligned}$$

To invert the Laplace transform we use a partial fractions expansion:

$$\begin{aligned}
\frac{3s+5}{(s+1)(s+2)} &= \frac{A}{s+1} + \frac{B}{s+2} \qquad \Rightarrow \qquad 3s+5 = A(s+2) + B(s+1) \\
s &= -1 \qquad \Rightarrow \qquad 2 = A(1) + 0 \qquad \Rightarrow \qquad A = 2 \\
s &= -2 \qquad \Rightarrow \qquad -1 = 0 + B(-1) \qquad \Rightarrow \qquad B = 1
\end{aligned}$$

So

$$\begin{aligned}
\mathcal{L}[y] &= \frac{3s+5}{(s+1)(s+2)} = \frac{2}{s+1} + \frac{1}{s+2} = 2\mathcal{L}\left[e^{-x}\right] + \mathcal{L}\left[e^{-2x}\right] = \mathcal{L}\left[2e^{-x} + e^{-2x}\right] \\
&\Rightarrow \qquad y = 2e^{-x} + e^{-2x}
\end{aligned}$$

□