

Math 2233
SOLUTIONS TO FINAL EXAM
8:s0 – 10:20 , May 11, 2001

1. (15 pts) Solve the following initial value problem.

$$xy' + 3y = 5x \quad , \quad y(1) = 3 \quad .$$

- This is a first order linear equation with $p(x) = 3/x$ and $g(x) = 5$

$$\begin{aligned} \mu &= \exp\left(\int p dx\right) = \exp\left(\int \frac{3}{x} dx\right) = \exp(3 \ln(x)) = x^3 \\ y &= \frac{1}{\mu} \int \mu g dx + \frac{C}{\mu} = x^{-3} \int 5x^3 + Cx^{-3} = x^{-3} \frac{5}{4} x^4 = \frac{5}{4} x + Cx^{-3} \\ 3 &= y(1) = \frac{5}{4} + C \quad \Rightarrow \quad C = \frac{7}{4} \\ \Rightarrow \quad y &= \frac{5}{4} x + \frac{7}{4x^3} \end{aligned}$$

□

2. (15 pts) Find an implicit solution of the following initial value problem.

$$(y/x + 4x) dx + (\ln(x) - 3) dy = 0 \quad , \quad y(1) = 1$$

(Hint: the equation is exact.)

- $$\begin{aligned} \Phi &= \int M \partial x + h_1(y) = \int (y/x + 4x) \partial x + h_1(y) = y \ln(x) + 2x^2 + h_1(y) \\ &= \int N \partial y + h_2(x) = \int (\ln(x) - 3) \partial y + h_2(x) = y \ln(x) - 3y + h_2(x) \\ \Rightarrow \quad h_1(y) &= -3y \quad , \quad h_2(x) = 2x^2 \quad \Rightarrow \quad \Phi(x, y) = y \ln(x) + 2x^2 - 3y \\ \Rightarrow \quad y \ln(x) + 2x^2 - 3y &= C \quad (\text{implicit solution}) \end{aligned}$$

□

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3.

(a)(5 pts) Show that the following differential equation is not exact.

$$y + (2x - ye^y) y' = 0$$

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$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} (y) = 1 \\ &\neq \frac{\partial N}{\partial x} = \frac{\partial}{\partial x} (2x - ye^y) = 2 \end{aligned}$$

□

(b) (10 pts) Find an integrating factor for the differential equation in (a). (Hint: look for an integrating factor that depends only on y .)

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$$\begin{aligned} F_2 &\equiv \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = \frac{1}{y} (2 - 1) = \frac{1}{y} \text{ does not depend on } x \\ \Rightarrow \mu(y) &= \exp \left(\int F_2(y) dy \right) = \exp \left(\int \frac{1}{y} dy \right) = \exp(\ln(y)) = y \end{aligned}$$

□

4. (10 pts) Find an implicit solution of the following ODE.

$$\frac{dy}{dx} = \frac{y^2 + yx}{x^2} .$$

(Hint: try the change of variables $z = y/x$.)

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$$\begin{aligned} y &= zx \Rightarrow y' = z'x + z \\ \Rightarrow z'x + z &= y' = \frac{y^2 + yx}{x^2} = \frac{(zx)^2 + (zx)x}{x^2} = z^2 + z \\ \Rightarrow z'x + z &= z^2 + z \Rightarrow z'x = z^2 \Rightarrow \frac{dz}{z^2} = \frac{dx}{x} \\ \Rightarrow \int \frac{dz}{z^2} - \int \frac{dx}{x} &= C \Rightarrow -\frac{1}{z} - \ln(x) = C \\ \Rightarrow z &= \frac{1}{-C - \ln(x)} \Rightarrow \frac{y}{x} = \frac{1}{-C - \ln(x)} \\ \Rightarrow y &= \frac{-x}{C + \ln(x)} \end{aligned}$$

□

5. (15 pts) Given that $y_1(x) = x^{-1}$ is one solution of $x^2y'' + 3xy' + y = 0$, use Reduction of Order to determine the general solution and the solution satisfying $y(1) = 1$, $y'(1) = 0$.

$$\begin{aligned}
 y_2 &= y_1 \int \frac{1}{(y_1)^2} \exp\left(-\int^x p ds\right) dx = x^{-1} \int x^2 \exp\left(\int^x -\frac{3}{s} ds\right) dx = x^{-1} \int x^2 \exp(-3 \ln(x)) dx \\
 &= x^{-1} \int x^2 x^{-3} dx = x^{-1} \int \frac{1}{x} dx = x^{-1} \ln(x) = x^{-1} \ln(x) \\
 \Rightarrow y(x) &= c_1 y_1(x) + c_2 y_2(x) = c_1 x^{-1} + c_2 x^{-1} \ln(x)
 \end{aligned}$$

□

6. (10 pts) Use the Method of Variation of Parameters to find the general solution of the following inhomogeneous differential equation.

$$y'' - 3y' + 2y = e^{3x} .$$

• First we find two linear independent solutions of the corresponding homogeneous problem

$$\begin{aligned}
 y'' - 3y' + 2y &= 0 \Rightarrow y = e^{\lambda x} \text{ with } \lambda \text{ satisfying } 0 = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2) \\
 \Rightarrow \lambda &= 1, 2 \Rightarrow y_1 = e^x, \quad y_2 = e^{2x}
 \end{aligned}$$

$$\begin{aligned}
 W[y_1, y_2] &= y_1 y_2' - y_1' y_2 = (e^x)(2e^{2x}) - (e^x)(e^{2x}) = (2 - 1)e^{3x} = e^{3x} \\
 y_p &= -y_1 \int \frac{y_2 g}{W[y_1, y_2]} dx + y_2 \int \frac{y_1 g}{W[y_1, y_2]} dx \\
 &= -e^x \int \frac{e^{2x} e^{3x}}{e^{3x}} dx + e^{2x} \int \frac{e^x e^{3x}}{e^{3x}} dx = -e^x \int e^{2x} dx + e^{2x} \int e^x dx \\
 &= -e^x \left[\frac{1}{2} e^{2x} \right] + e^{2x} [e^x] = \frac{1}{2} e^{3x} \\
 \Rightarrow y &= y_p + c_1 y_1 + c_2 y_2 = \frac{1}{2} e^{3x} + c_1 e^x + c_2 e^{2x}
 \end{aligned}$$

□

7. (10 pts) What is the minimal radius of convergence of a power series solution of

$$(1 + x^2)y'' + 2y' + xy = 0$$

about $x_0 = 2$?

- The functions $p(x) = 2/(1 + x^2)$ and $q(x) = x/(1 + x^2)$ have denominators that vanish when $x = \pm i$. These singular points have coordinates $(0, \pm 1)$ in the complex plane and so are a distance

$$D = \sqrt{(2 - 0)^2 + (0 \mp 1)^2} = \sqrt{5}$$

from the point $(2, 0)$ (corresponding to the expansion point $x_0 = 2$). Thus,

$$R \geq \sqrt{5}$$

□

8. (15 pts) Find the recursion relations for a power series solution about $x_0 = 1$ for the following differential equation.

$$xy'' - 2y = 0$$

- We set $y = \sum_{n=0}^{\infty} a_n (x - 1)^n$ and demand

$$\begin{aligned} 0 &= xy'' - 2y = x \sum_{n=0}^{\infty} n(n-1) a_n (x-1)^{n-2} - 2 \sum_{n=0}^{\infty} a_n (x-1)^n \\ &= [(x-1) + 1] \sum_{n=0}^{\infty} n(n-1) a_n (x-1)^{n-2} + \sum_{n=0}^{\infty} -2a_n (x-1)^n \\ &= \sum_{n=0}^{\infty} n(n-1) a_n (x-1)^{n-1} + \sum_{n=0}^{\infty} n(n-1) a_n (x-1)^{n-2} + \sum_{n=0}^{\infty} -2a_n (x-1)^n \\ &= \sum_{n=-1}^{\infty} (n+1)(n) a_{n+1} (x-1)^n + \sum_{n=-2}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n + \sum_{n=0}^{\infty} -2a_n (x-1)^n \\ &= 0 + \sum_{n=0}^{\infty} (n+1)(n) a_{n+1} (x-1)^n + 0 + 0 + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n + \sum_{n=0}^{\infty} -2a_n (x-1)^n \\ &= \sum_{n=0}^{\infty} [n(n+1) a_{n+1} + (n+2)(n+1) a_{n+2} - 2a_n] (x-1)^n \\ \Rightarrow \quad a_{n+2} &= \frac{2a_n - n(n+1) a_{n+1}}{(n+2)(n+1)}, \quad n = 0, 1, 2, 3, \dots \end{aligned}$$

□

9. (15 pts) Given that the recursion relations for $y'' - xy' + y = 0$ about $x_0 = 0$ are

$$a_{n+2} = \frac{(n-1)a_n}{(n+2)(n+1)}, \quad n = 0, 1, 2, 3, \dots$$

Write down the first 4 terms (i.e., to order x^3) for the power series solution satisfying $y(0) = 1$, $y'(0) = 2$.

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$$\begin{aligned} y(0) &= 1 \Rightarrow a_0 = 1 \\ y'(0) &= 2 \Rightarrow a_1 = 2 \\ a_2 &= a_{0+2} = \frac{(0-1)a_0}{(0+2)(0+1)} = -\frac{a_0}{2} = -\frac{1}{2} \\ a_3 &= a_{1+3} = \frac{(1-1)a_1}{(1+2)(1+1)} = 0a_1 = 0 \\ y(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + \dots \\ &= 1 + 2x - \frac{1}{2}x^2 + 0 + \dots \end{aligned}$$

□

10. (15 pts) Consider the following linear differential equation $2xy'' - 2y = 0$. Assume a solution of the form $y = x^r \sum_{n=0}^{\infty} a_n x^n$ and determine the possible values of r .

• We set $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ and demand

$$\begin{aligned} 0 &= 2xy'' - 2y = 2x \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} - 2 \sum_{n=0}^{\infty} a_n x^{n+r} \\ &= \sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty} -2a_n x^{n+r} \\ &= \sum_{n=-1}^{\infty} 2(n+r+1)(n+r)a_{n+1} x^{n+r} + \sum_{n=0}^{\infty} -2a_n x^{n+r} \\ &= 2(-1+r+1)(-1+r)a_{-1+1} x^{-1+r} + \sum_{n=0}^{\infty} 2(n+r+1)(n+r)a_{n+1} x^{n+r} + \sum_{n=0}^{\infty} -2a_n x^{n+r} \\ &= 2r(r-1)a_0 x^{r-1} + \sum_{n=0}^{\infty} [2(n+r+1)(n+r)a_{n+1} - 2a_n] x^{n+r} \end{aligned}$$

So we need

$$0 = 2r(r-1) \Rightarrow r = 0, 1$$

□

11. (15 pts) Use the Laplace Transform technique to solve the following initial value problem.

$$y'' - 2y' - 3y = 0, \quad y(0) = 3, \quad y'(0) = 1$$

$$\begin{aligned}
0 &= \mathcal{L}[y'' - 2y' - 3y] = \mathcal{L}[y''] - 2\mathcal{L}[y'] - 3\mathcal{L}[y] \\
&= (s^2\mathcal{L}[y] - sy(0) - y'(0)) - 2(s\mathcal{L}[y] - y(0)) - 3\mathcal{L}[y] \\
&= s^2\mathcal{L}[y] - 3s - 1 - 2s\mathcal{L}[y] + 6 - 3\mathcal{L}[y] \\
&= (s^2 - 2s - 3)\mathcal{L}[y] - 3s + 5 \\
\Rightarrow \mathcal{L}[y] &= \frac{3s - 5}{s^2 - 2s - 3} = \frac{3s - 5}{(s - 3)(s + 1)}
\end{aligned}$$

We use a Partial Fractions expansion to invert the Laplace transform of $(3s - 5) / [(s - 3)(s + 1)]$

$$\begin{aligned}
\frac{3s - 5}{(s - 3)(s + 1)} &= \frac{A}{s - 3} + \frac{B}{s + 1} \Rightarrow 3s - 5 = (s + 1)A + (s - 3)B \\
s = -1 &\Rightarrow -8 = 0 - 4B \Rightarrow B = 2 \\
s = 3 &\Rightarrow 4 = 4A + 0 \Rightarrow A = 1
\end{aligned}$$

So

$$\begin{aligned}
\mathcal{L}[y] &= \frac{1}{s - 3} + 2\frac{1}{s + 1} = \mathcal{L}[e^{3x}] + 2\mathcal{L}[e^{-x}] = \mathcal{L}[e^{3x} + 2e^{-x}] \\
\Rightarrow y &= e^{3x} + 2e^{-x}
\end{aligned}$$

□