1. Compute the Laplace transform of the following functions.

(a) \( f(t) = t \)

- Let \( f(t) = t \).

\[
\mathcal{L}[f] = \int_0^\infty te^{-st} \, dt
\]

Integrating by parts, with

\[
\begin{align*}
  u &= t & \quad & dv = e^{-st} \, dt \\
  du &= dt & \quad & v = -\frac{1}{s}e^{-st}
\end{align*}
\]

we get

\[
\int_0^\infty te^{-st} \, dt = \int_0^\infty vdu
\]

\[
= uv|_0^\infty - \int_0^\infty vdu
\]

\[
= (t) \left( -\frac{1}{s}e^{-st} \right)|_0^\infty - \int_0^\infty \left( -\frac{1}{s}e^{-st} \right) \, dt
\]

\[
= 0 - 0 - \frac{1}{s^2}e^{-st}|_0^\infty
\]

\[
= \frac{1}{s^2}
\]

\[
\mathcal{L}[f] = \frac{1}{s^2}
\]

(b) \( f(t) = t^n \)

- Now let \( f(t) = t^n \).

\[
\mathcal{L}[f] = \int_0^\infty t^n e^{-st} \, dt
\]

Integrating by parts, with

\[
\begin{align*}
  u &= t^n & \quad & dv = e^{-st} \, dt \\
  du &= nt^{n-1} \, dt & \quad & v = -\frac{1}{s}e^{-st}
\end{align*}
\]
we get
\[\int_0^\infty t^n e^{-t} dt = \int_0^\infty vdu\]
\[= uv|_0^\infty - \int_0^\infty vdu\]
\[= (t^n) \left( \frac{1}{s} e^{-st} \right)|_0^\infty - \int_0^\infty \left( \frac{1}{s} e^{-st} \right) (nt^{n-1} dt)\]
\[= 0 - 0 + \frac{n}{s} \int_0^\infty t^{n-1} e^{-st} dt\]
\[= \frac{n}{s} L \left[ t^{n-1} \right]\]
\[= \frac{n(n-1)}{s} L \left[ t^{n-2} \right]\]
\[= \frac{n(n-1)(n-2)}{s^3} L \left[ t^{n-3} \right]\]
\[= \cdots \]
\[= \frac{n(n-2) \cdots (2)}{s^{n-1}} L \left[ t \right]\]
\[= \frac{n!}{s^{n+1}}\]

2. Use the formula
\[\sin(bt) = \frac{e^{ibt} - e^{-ibt}}{2i}\]
to compute the Laplace transform of \(\sin(bt)\).

• Let
\[f(t) = \sin(bt) = \frac{1}{2i} \left( e^{ibt} - e^{-ibt} \right)\]
Then
\[L \left[ \sin(bt) \right] = \int_0^\infty \frac{1}{2i} \left( e^{ibt} - e^{-ibt} \right) e^{-st} dt\]
\[= \frac{1}{2i} \int_0^\infty \left( e^{-s+ibt} - e^{-s-ibt} \right) dt\]
\[= \frac{1}{2i} \left( \frac{1}{s+ibt} - \frac{1}{s-ibt} \right)\]
\[= \frac{1}{2i} \left( \frac{s+ibt - (s-ibt)}{s^2 + b^2} \right)\]
\[= \frac{b}{s^2 + b^2}\]

3. Use the Laplace transform to solve the given initial value problem
\[y'' - y' - 6y = 0 \quad ; \quad y(0) = 1 \quad , \quad y'(0) = -1\]
Taking the Laplace transform of both sides of the differential equation yields

\[ 0 = \mathcal{L}[y''] - \mathcal{L}[y'] - \mathcal{L}[6y] \]
\[ = (s^2 \mathcal{L}[y] - sy(0) - y'(0)) - (s\mathcal{L}[y] - y(0)) - 6\mathcal{L}[y] \]
\[ = s^2 \mathcal{L}[y] - s(1) - (-1) - s\mathcal{L}[y] + (1) - 6\mathcal{L}[y] \]
\[ = (s^2 - s - 6) \mathcal{L}[y] - s + 2 \]

or

\[ \mathcal{L}[y] = \frac{s - 2}{s^2 - s - 6} = \frac{s - 2}{(s + 2)(s - 3)} \]

the differential equation for \( y \) becomes an algebraic equation for \( \mathcal{L}[y] \). To undo this Laplace transform we first carry out a partial fractions expansion of the right hand side of the equation for \( \mathcal{L}[y] \).

\[ \frac{s - 2}{(s + 2)(s - 3)} = \frac{A}{s + 2} + \frac{B}{s - 3} \Rightarrow s - 2 = A(s - 3) + B(s + 2) \]

This expansion must be valid for all values of \( s \); in particular when \( s = -2 \) and when \( s = 3 \). In the former case we have

\[ s = -2 \Rightarrow -4 = (-2) - 2 = A(-2 - 3) + B(-2 + 2) = -5A \]

so we must have \( A = \frac{1}{5} \). In the latter case, we have

\[ s = 3 \Rightarrow 1 = (3) - 2 = A(3 - 3) + B(3 + 2) = 5B \]

so \( B = \frac{1}{5} \). We then have

\[ \mathcal{L}[y] = \frac{s - 2}{(s + 2)(s - 3)} = \frac{4}{5} \frac{1}{s + 2} + \frac{1}{5} \frac{1}{s - 3} \]
\[ = \mathcal{L} \left[ \frac{4}{5} e^{-2x} + \frac{1}{5} e^{3x} \right] \]

Hence, (taking inverse Laplace transform of both sides)

\[ y = \frac{4}{5} e^{-2x} + \frac{1}{5} e^{3x} \]

4. Use the Laplace transform to solve the given initial value problem

\[ y'' - 2y' + 2y = 0 \quad ; \quad y(0) = 0 , \quad y'(0) = 1 \]

Taking the Laplace transform of both sides of the differential equation yields

\[ 0 = s^2 \mathcal{L}[y] - sy(0) - y'(0) - 2 (s\mathcal{L}[y] - y(0)) + 2\mathcal{L}[y] \]
\[ = (s^2 - 2s + 2) \mathcal{L}[y] - 1 \]

or

\[ \mathcal{L}[y] = \frac{1}{s^2 - 2s + 2} = \frac{1}{s^2 - 2s + 1 + 1} = \frac{1}{(s - 1)^2 + 1} \]

We now consult a table of Laplace transform and spot the following identity

\[ \mathcal{L} \left[ e^{at} \sin(bt) \right] = \frac{b}{(s - a)^2 + b^2} \]
which looks just like the right hand side of our expression for \( \mathcal{L}[y] \) once we take \( a = -1 \) and \( b = 1 \). We conclude

\[
\mathcal{L}[y] = \mathcal{L}[e^{-x} \sin(x)]
\]

or

\[
y(x) = e^{-x} \sin(x)
\]

5. Use the Laplace transform to solve the given initial value problem

\[
y'' - 2y' - 2y = 0 \quad ; \quad y(0) = 2 \quad ; \quad y'(0) = 0
\]

- Taking the Laplace transform of the differential equation we get

\[
0 = s^2 \mathcal{L}[y] - sy(0) - y'(0) - 2 (s \mathcal{L}[y] - y(0)) - 2 \mathcal{L}[y]
\]

\[
= (s^2 - 2s - 2) \mathcal{L}[y] - 2s + 2
\]

Thus,

\[
\mathcal{L}[y] = \frac{2s - 2}{s^2 - 2s - 2} = \frac{2}{s - 1} + \frac{1 - 3}{(s - 1)^2 - 3}
\]

To undo this Laplace transform it is helpful to first look at a table of Laplace transforms. From such a table one finds

\[
\mathcal{L}[e^{at} \cosh(bt)] = \frac{s - a}{(s - a)^2 - b^2}
\]

\[
\mathcal{L}[e^{at} \sinh(bt)] = \frac{b}{(s - a)^2 - b^2}
\]

and so taking \( a = 1 \) and \( b = \sqrt{3} \) we have

\[
\mathcal{L}[y] = 2 \mathcal{L}[e^{\sqrt{3}x} \cosh(\sqrt{3}x)] - \frac{2 + 2\sqrt{3}}{\sqrt{3}} \mathcal{L}[e^{\sqrt{3}x} \sinh(\sqrt{3}x)]
\]

\[
= \mathcal{L} \left[ 2e^{\sqrt{3}x} \cosh(\sqrt{3}x) - \frac{2 + 2\sqrt{3}}{\sqrt{3}} e^{\sqrt{3}x} \sinh(\sqrt{3}x) \right]
\]

so

\[
y(x) = 2e^{\sqrt{3}x} \cosh(\sqrt{3}x) - \frac{2 + 2\sqrt{3}}{\sqrt{3}} e^{\sqrt{3}x} \sinh(\sqrt{3}x)
\]

6. Use the Laplace transform to solve the given initial value problem

\[
y'' + 2y' + y = 4e^{-t} \quad ; \quad y(0) = 2 \quad ; \quad y'(0) = -1
\]

- Taking the Laplace transform of both sides of the differential equation we get

\[
s^2 \mathcal{L}[y] - sy(0) - y'(0) + 2 (s \mathcal{L}[y] - y(0)) + \mathcal{L}[y] = \mathcal{L}[4e^{-t}]
\]
or

\((s^2 + 2s + 1) \mathcal{L}[y] = 2s + 1 - 4 = \frac{4}{s + 1}\)

or

\((s + 1)^2 \mathcal{L}[y] = \frac{4}{s + 1} + 2s + 3 = \frac{4 + 2s^2 + 3s + 3}{s + 1} = \frac{2s^2 + 3s + 7}{s + 1}\)

or

\[\mathcal{L}[y] = \frac{2s^2 + 3s + 7}{(s + 1)^3}\]

We now determine the partial fractions expansion of the right hand side. The general ansatz is

\[\frac{P(x)}{(s + a)^3} = \frac{A}{s + a} + \frac{B}{(s + a)^2} + \frac{C}{(s + a)^3}\]

and so we will try to find constants \(A, B, C\) such that

\[\frac{2s^2 + 5s + 7}{(s + 1)^3} = \frac{A}{s + 1} + \frac{B}{(s + 1)^2} + \frac{C}{(s + 1)^3}\]

Multiplying both sides by \((s + 1)^3\) we get

\[2s^2 + 5s + 7 = A (s + 1)^2 + B (s + 1) + C\]

Plugging in \(s = -1\) we find

\[2 - 5 + 7 = C\]

or

\[C = 4\]

Plugging in \(s = 0\) yields

\[7 = A + B + C = A + B + 4\]

or

\[A + B = 3\]

Plugging in \(s = 1\) yields

\[14 = 4A + 2B + C = 4A + 2B + 4\]

or

\[4A + 2B = 10\]

or

\[2A + B = 5\]

We now solve

\[\begin{align*}
A + B &= 3 \\
2A + B &= 5
\end{align*}\]

for \(A\) and \(B\). Subtracting the first equation from the second we obtain

\[A + 0 = 2 \quad \Rightarrow \quad A = 2\]

Now the first equation yields

\[2 + B = 3 \quad \Rightarrow \quad B = 1\]

Thus, \(A = 2\), \(B = 1\), and \(C = 4\). Applying this partial fractions expansion to the equation for \(\mathcal{L}[y]\) now yields

\[\mathcal{L}[y] = \frac{2s^2 + 3s + 7}{(s + 1)^3} = \frac{2}{s + 1} + \frac{1}{(s + 1)^2} + \frac{4}{(s + 1)^3}\]
Now from a Table of Laplace transforms we find

\[ \mathcal{L} \left[ t^n e^{at} \right] = \frac{n!}{(s+a)^{n+1}} \]

Hence

\[ \frac{1}{s+1} = \mathcal{L}[e^{-t}] \]
\[ \frac{1}{(s+1)^2} = \mathcal{L}[te^{-t}] \]
\[ \frac{1}{(s+1)^3} = \frac{1}{2} \mathcal{L}[t^2 e^{-t}] \]

so

\[ \mathcal{L}[y] = 2\mathcal{L}[e^{-t}] + \mathcal{L}[te^{-t}] + 4\frac{1}{2} \mathcal{L}[t^2 e^{-t}] \]
\[ = \mathcal{L} \left[ 2e^{-t} + te^{-t} + 2t^2 e^{-t} \right] \]

Taking the inverse Laplace transform of both sides we finally get

\[ y(t) = (2t^2 + t + 2) e^{-t}. \]