Math 2233
Homework Set 8

1. Determine the lower bound for the radius of convergence of series solutions about each given point x_o.

(a) $y'' + 4y' + 6xy = 0$, $x_0 = 0$

- Since the coefficient functions

$$\begin{aligned} p(x) &= 4 \\ q(x) &= 6x \end{aligned}$$

are perfectly analytic for all x, the differential equation thus possesses no singular points. Thus, every power series solution

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_o)^n$$

will converges for all x and all x_o. In particular, the radius of convergence for solutions about $x_o = 0$ will be infinite. □

(b) $(x-1)y'' + xy' + 6xy = 0$, $x_0 = 4$

- Since the coefficient functions

$$\begin{aligned} p(x) &= \frac{x}{x-1} \\ q(x) &= \frac{6x}{x-1} \end{aligned}$$

are both undefined for $x = 1$. Therefore, $x = 1$ is a singular point for this differential equation. According to the theorem stated in lecture, if

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_o)^n$$

is a power series solution, then its radius of convergence will be at least as large as the distance (in the complex plane) from the expansion point x_o and the closest singularity of the functions $p(x)$ and $q(x)$. In the case at hand, $x_0 = 4$ and the closest (in fact, the only) singular point of the coefficient functions $p(x)$ and $q(x)$ is $x = 1$. Since

$$\|4 - 1\| = 3$$

we can conclude that the radius of convergence of a series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x-4)^n$$

will be at least 3. In other words, the series solution will be valide for all x in the interval

$$|x - 4| < 3$$

or, equivalently, for all x such that

$$1 < x < 7$$

□

(c) $\left(4 + x^2\right) y'' + 4xy' + y = 0$, $x_0 = 0$

- In this case, the coefficient functions

$$\begin{aligned} p(x) &= \frac{4x}{4+x^2} \\ q(x) &= \frac{1}{4+x^2} \end{aligned}$$

both have singularities when

$$4 + x^2 = 0 \quad \Rightarrow \quad x = \pm 2i$$

These two singularies correspond to the points $(0, \pm 2)$ when we represent points in the complex plane as points in the two dimenional plane. Under this representation of the complex plane, the expansion point $x_0 = 0$ corresponds to the point $(0, 0)$. Therefore the distances between the expansion point and the singularity are

$$\begin{aligned} dist(2i, 0) &= \sqrt{(0-0)^2 + (2-0)^2} = 2 \\ dist(-2i, 0) &= \sqrt{(0-0)^2 + (-2-0)^2} = 2 \end{aligned}$$

Hence, the minimal distance is 2, and so the radius of convergence of a power series solution about 0 is at least 2. □

(d) $\left(1+x^2\right) y'' + 4xy' + y = 0$, $x_0 = 2$

- In this case the coefficient functions

$$\begin{aligned} p(x) &= \frac{4x}{1+x^2} \\ q(x) &= \frac{1}{1+x^2} \end{aligned}$$

both have singularities when

$$1 + x^2 = 0 \ \Rightarrow \ x = \pm i$$

These two singularies correspond to the points $(0, \pm 1)$ when we represent points in the complex plane as points in the two dimenional plane. Under this representation of the complex plane, the expansion point $x_0 = 2$ corresponds to the point $(2, 0)$. Therefore the distances between the expansion point and the singularity are

$$\begin{aligned} dist(i, 2) &= \sqrt{(0-2)^2 + (1-0)^2} = \sqrt{5} \\ dist(-i, 2) &= (0-2)^2 + (-1-0)^2 = \sqrt{5} \end{aligned}$$

Hence, the distance between the expansion point and the closest singularity is $\sqrt{5}$ and so the radius of convergence of a power series solution about the point $x_o = 2$ will be at least $\sqrt{5}$. □

2. Determine the singular points of the following differential equations and state whether they are regular or irregular singular points.

(a) $xy'' + (1-x)y' + xy = 0$

- In this case, the coefficient functions are

$$\begin{aligned} p(x) &= \frac{1-x}{x} \\ q(x) &= 1 \end{aligned}$$

Since $p(x)$ is undefined for $x = 0$, 0 is a singular point. Since the limits

$$\begin{aligned} \lim_{x\to 0}(x-0)p(x) &= \lim_{x\to 0}(1-x) = 1 \\ \lim_{x\to 0}(x-0)^2 q(x) &= \lim_{x\to 0} x^3 = 0 \end{aligned}$$

both exist, $x = 0$ is a regular singular point. Alternatively, one could say that because the degree of the singularity of the function $p(x)$ at the point $x = 0$ is less than or equal to 1 and the degree of the singularity of the function $q(x)$ is less than or equal to 2, we have regular singular point at $x = 0$. □

(b) $x^2(1-x)^2 y'' + 2xy + 4y = 0$

- In this case, the coefficient functions are

$$\begin{aligned} p(x) &= \frac{2}{x(1-x)^2} \\ q(x) &= \frac{4}{x^2(1-x)^2} \end{aligned}$$

The function $p(x)$ evidently has a singularity of degree 1 at $x = 0$ and a singularity of degree 2 at $x = 1$. The function $q(x)$ has singularities of degree 2 at $x = 0$ and $x = 1$. In order to be a regularity singular point the degree of the singularity of $p(x)$ must not exceed 1 and the degree of the singularity of $q(x)$ must not exceed 2. Therefore, $x = 0$ is a regular singular point and $x = 1$ is an irregular singular point.. □

(c) $(1-x^2)^2 y'' + x(1-x)y' + (1+x)y = 0$

- In this case,the coefficient functions are

$$\begin{aligned} p(x) &= \frac{x(1-x)}{(1-x^2)^2} = \frac{x(1-x)}{(1-x)^2(1+x)^2} = \frac{x}{(1-x)(1+x)^2} \\ q(x) &= \frac{4}{x^2(1-x)^2} = \frac{1+x}{(1-x)^2(1+x)^2} = \frac{1}{(1-x)^2(1+x)} \end{aligned}$$

The function $p(x)$ evidently has a singularity of degree 1 at $x = 1$ and a singularity of degree 2 at $x = -1$. The function $q(x)$ has a singularity of degree 1 at $x = 1$ and a singularity of degree 2 at $x = -1$. In order to be a regularity singular point the degree of the singularity of $p(x)$ must not exceed 1 and the degree of the singularity of $q(x)$ must not exceed 2. Therefore, $x = 1$ is a regular singular point and $x = -1$ is an irregular singular point. □

3. Compute the Laplace transform of the following functions.

(a) $f(t) = t$

- Let $f(t) = t$.

$$\mathcal{L}[f] = \int_0^\infty te^{-t}dt$$

Integrating by parts, with

$$\begin{aligned} u &= t & dv &= e^{-st}dt \\ du &= dt & v &= -\tfrac{1}{s}e^{-st} \end{aligned}$$

we get

$$\begin{aligned} \int_0^\infty te^{-t}dt &= \int_0^\infty vdu \\ &= uv|_0^\infty - \int_0^\infty vdu \\ &= (t)\left(-\frac{1}{s}e^{-st}\right)\Big|_0^\infty - \int_0^\infty \left(-\frac{1}{s}e^{-st}\right)dt \\ &= 0 - 0 - \frac{1}{s^2}e^{-st}\Big|_0^\infty \\ &= \frac{1}{s^2} \end{aligned}$$

□

(b) $f(t) = t^n$

- Now let $f(t) = t^n$.

$$\mathcal{L}[f] = \int_0^\infty t^n e^{-t}dt$$

Integrating by parts, with

$$\begin{aligned} u &= t^n & dv &= e^{-st}dt \\ du &= nt^{n-1}dt & v &= -\tfrac{1}{s}e^{-st} \end{aligned}$$

we get

$$\begin{aligned}
\int_0^\infty t^n e^{-t} dt &= \int_0^\infty v du \\
&= uv|_0^\infty - \int_0^\infty v du \\
&= (t^n)\left(-\frac{1}{s}e^{-st}\right)\Big|_0^\infty - \int_0^\infty \left(-\frac{1}{s}e^{-st}\right)\left(nt^{n-1}dt\right) \\
&= 0 - 0 + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt \\
&= \frac{n}{s}\mathcal{L}\left[t^{n-1}\right] \\
&= \frac{n}{s}\frac{n-1}{s}\mathcal{L}\left[t^{n-2}\right] \\
&= \frac{n(n-1)(n-2)}{s^3}\mathcal{L}\left[t^{n-3}\right] \\
&\vdots \\
&= \frac{n(n-2)\cdots(2)}{s^{n-1}}\mathcal{L}\left[t\right] \\
&= \frac{n!}{s^{n+1}}
\end{aligned}$$

□

4. Use the Laplace transform to solve the given initial value problem.

(1) $$y'' - y' - 6y = 0 \quad ; \quad y(0) = 1 \quad , \quad y'(0) = -1$$

- Taking the Laplace transform of both sides of the differential equation yields

$$\begin{aligned}
0 &= \mathcal{L}[y''] - \mathcal{L}[y'] - \mathcal{L}[6y] \\
&= \left(s^2\mathcal{L}[y] - sy(0) - y'(0)\right) - \left(s\mathcal{L}[y] - y(0)\right) - 6\mathcal{L}[y] \\
&= s^2\mathcal{L}[y] - s(1) - (-1) - s\mathcal{L}[y] + (1) - 6\mathcal{L}[y] \\
&= \left(s^2 - s - 6\right)\mathcal{L}[y] - s + 2
\end{aligned}$$

or

$$\mathcal{L}[y] = \frac{s-2}{s^2 - s - 6} = \frac{s-2}{(s+2)(s-3)}$$

the differential equation for y becomes an algebraic equation for $\mathcal{L}[y]$. To undo this Laplace transform we first carry out a partial fractions expansion of the right hand side of the equation for $\mathcal{L}[y]$.

$$\frac{s-2}{(s+2)(s-3)} = \frac{A}{s+2} + \frac{B}{s-3} \Rightarrow \quad s - 2 = A(s-3) + B(s+2)$$

This expansion must be valid for all values of s; in particular when $s = -2$ and when $s = 3$. In the former case
we have

$$s = -2 \quad \Rightarrow \quad -4 = (-2) - 2 = A(-2-3) + B(-2+2) = -5A$$

so we must have $A = \frac{4}{5}$. In the latter case, we have

$$s = 3 \quad \Rightarrow \quad 1 = (3) - 2 = A(3-3) + B(3+2) = 5B$$

so $B = \frac{1}{5}$. We then have

$$\begin{aligned}
\mathcal{L}[y] &= \frac{s-2}{(s+2)(s-3)} \\
&= \frac{4}{5}\frac{1}{s+2} + \frac{1}{5}\frac{1}{s-3} \\
&= \frac{4}{5}\mathcal{L}[e^{-2x}] + \frac{1}{5}\mathcal{L}[e^{3x}] \\
&= \mathcal{L}\left[\frac{4}{5}e^{-2x} + \frac{1}{5}e^{3x}\right]
\end{aligned}$$

Hence, (taking inverse Laplace transform of both sides)

$$y = \frac{4}{5}e^{-2x} + \frac{1}{5}e^{3x}$$

$\square$

5. Use the Laplace transform to solve the given initial value problem.

$$y'' - 2y' + 2y = 0 \quad ; \quad y(0) = 0 \quad , \quad y'(0) = 1 \quad .$$

- Taking the Laplace transform of both sides of the differential equation yields

$$\begin{aligned}
0 &= s^2\mathcal{L}[y] - sy(0) - y'(0) - 2\left(s\mathcal{L}[y] - y(0)\right) + 2\mathcal{L}[y] \\
&= \left(s^2 - 2s + 2\right)\mathcal{L}[y] - 1
\end{aligned}$$

or

$$\mathcal{L}[y] = \frac{1}{s^2 - 2s + 2} = \frac{1}{s^2 - 2s + 1 + 1} = \frac{1}{(s-1)^2 + 1}$$

We now consult a table of Laplace transform and spot the following identity

$$\mathcal{L}\left[e^{at}\sin(bt)\right] = \frac{b}{(s-a)^2 + b^2}$$

which looks just like the right hand side of our expression for $\mathcal{L}[y]$ once we thake $a = -1$ and $b = 1$. We conclude

$$\mathcal{L}[y] = \mathcal{L}[e^{-x}\sin(x)]$$

or

$$y(x) = e^{-x}\sin(x)$$

$\square$

6. Use the Laplace transform to solve the given initial value problem.

$$y'' - 2y' - 2y = 0 \quad ; \quad y(0) = 2 \quad , \quad y'(0) = 0 \quad .$$

- Taking the Laplace transform of the differential equation we get

$$\begin{aligned}
0 &= s^2\mathcal{L}[y] - sy(0) - y'(0) - 2\left(s\mathcal{L}[y] - y(0)\right) - 2\mathcal{L}[y] \\
&= \left(s^2 - 2s - 2\right)\mathcal{L}[y] - 2s + 2 \\
&= \left(s^2 - 2s - 2\right)\mathcal{L}[y] - 2s + 2
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathcal{L}[y] &= \frac{2s-2}{s^2-2x-2} \\
&= 2\frac{s-1}{s^2-2s+1-3} \\
&= 2\frac{s-\sqrt{3}-1+\sqrt{3}}{(s-1)^2-3} \\
&= 2\frac{s-\sqrt{3}}{(s-1)^2-3} - \frac{2+2\sqrt{3}}{\sqrt{3}}\frac{\sqrt{3}}{(s-1)^2-3}
\end{aligned}$$

To undo this Laplace transform it is helpful to first look at a table of Laplace transforms. From such a table one finds

$$\begin{aligned}
\mathcal{L}[e^{at}\cosh(bt)] &= \frac{s-a}{(s-a)^2-b^2} \\
\mathcal{L}[e^{at}\sinh(bt)] &= \frac{b}{(s-a)^2-b^2}
\end{aligned}$$

and so taking $a=1$ and $b=\sqrt{3}$ we have

$$\begin{aligned}
\mathcal{L}[y] &= 2\mathcal{L}[e^x\cosh(\sqrt{3}x)] - \frac{2+2\sqrt{3}}{\sqrt{3}}\mathcal{L}\left[e^x\sinh(\sqrt{3}x)\right] \\
&= \mathcal{L}\left[2e^x\cosh(\sqrt{3}x) - \frac{2+2\sqrt{3}}{\sqrt{3}}e^x\sinh(\sqrt{3}x)\right]
\end{aligned}$$

so

$$y(x) = 2e^x\cosh(\sqrt{3}x) - \frac{2+2\sqrt{3}}{\sqrt{3}}e^x\sinh(\sqrt{3}x)$$

$\square$

7. Use the Laplace transform to solve the given initial value problem.

$$y''+2y'+y=4e^{-t} \qquad ; \qquad y(0)=2 \quad , \quad y'(0)=-1 \quad .$$

- Taking the Laplace transform of both sides of the differential equation we get

$$s^2\mathcal{L}[y]-sy(0)-y'(0)+2\left(s\mathcal{L}[y]-y(0)\right)+\mathcal{L}[y]=\mathcal{L}[4e^{-t}]$$

or

$$\left(s^2+2s+1\right)\mathcal{L}[y]-2s+1-4=\frac{4}{s+1}$$

or

$$(s+1)^2\mathcal{L}[y]=\frac{4}{s+1}+2s+3=\frac{4+2s^2+3s+3}{s+1}=\frac{2s^2+3s+7}{s+1}$$

or

$$\mathcal{L}[y]=\frac{2s^2+3s+7}{(s+1)^3}$$

We now determine the partial fractions expansion of the right hand side. The general *ansatz* is

$$\frac{P(x)}{(s+a)^3}=\frac{A}{s+a}+\frac{B}{(s+a)^2}+\frac{C}{(s+a)^3}$$

and so we will try to find constants A,B,C such that

$$\frac{2s^2+5s+7}{(s+1)^3}=\frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{C}{(s+1)^3}.$$

Multiplying both sides by $(s+1)^3$ we get

$$2s^2+5s+7=A\,(s+1)^2+B\,(s+1)+C \quad .$$

Plugging in $s=-1$ we find

$$2-5+7=C$$

or

$$C=4.$$

Plugging in $s=0$ yields

$$7=A+B+C=A+B+4$$

or

$$A+B=3.$$

Plugging in $s=1$ yields

$$14=4A+2B+C=4A+2B+4$$

or

$$4A+2B=10$$

or

$$2A+B=5.$$

We now solve

$$\begin{aligned} A+B &= 3 \\ 2A+B &= 5 \end{aligned}$$

for A and B. Subtracting the first equation from the second we obtain

$$A+0=2 \qquad \Rightarrow \qquad A=2.$$

Now the first equation yields

$$2+B=3 \qquad \Rightarrow \qquad B=1.$$

Thus, $A=2$, $B=1$, and $C=4$. Applying this partial fractions expansion to the equation for $\mathcal{L}[y]$ now yields

$$\mathcal{L}[y]=\frac{2s^2+3s+7}{(s+1)^3}=\frac{2}{s+1}+\frac{1}{(s+1)^2}+\frac{4}{(s+1)^3}$$

Now from a Table of Laplace transforms we find

$$\mathcal{L}\left[t^n e^{at}\right]=\frac{n!}{(s+a)^{n+1}}$$

Hence

$$\begin{aligned} \frac{1}{s+1} &= \mathcal{L}[e^{-t}] \\ \frac{1}{(s+1)^2} &= \mathcal{L}[te^{-t}] \\ \frac{1}{(s+1)^3} &= \frac{1}{2}\mathcal{L}[t^2e^{-t}] \end{aligned}$$

so

$$\begin{aligned} \mathcal{L}[y] &= 2\mathcal{L}[e^{-t}]+1\mathcal{L}[te^{-t}]+4\frac{1}{2}\mathcal{L}[t^2e^{-t}] \\ &= \mathcal{L}\left[2e^{-t}+te^{-t}+2t^2e^{-t}\right] \end{aligned}$$

Taking the inverse Laplace transform of both sides we finally get

$$y(t)=\left(2t^2+t+2\right)e^{-t}.$$

□