1. Determine the lower bound for the radius of convergence of series solutions about each given point $x_o$.

(a) $y'' + 4y' + 6xy = 0$, $x_0 = 0$

- Since the coefficient functions
  
  \[ p(x) = 4 \]
  
  \[ q(x) = 6x \]

  are perfectly analytic for all $x$, the differential equation thus possesses no singular points. Thus, every power series solution

  \[ y(x) = \sum_{n=0}^{\infty} a_n (x - x_o)^n \]

  will converges for all $x$ and all $x_o$. In particular, the radius of convergence for solutions about $x_o = 0$ will be infinite.

(b) $(x - 1)y'' + xy' + 6xy = 0$, $x_0 = 4$

- Since the coefficient functions
  
  \[ p(x) = \frac{x}{x - 1} \]
  
  \[ q(x) = \frac{6x}{x - 1} \]

  are both undefined for $x = 1$. Therefore, $x = 1$ is a singular point for this differential equation. According to the theorem stated in lecture, if

  \[ y(x) = \sum_{n=0}^{\infty} a_n (x - x_o)^n \]

  is a power series solution, then its radius of convergence will be at least as large as the distance (in the complex plane) from the expansion point $x_o$ and the closest singularity of the functions $p(x)$ and $q(x)$. In the case at hand, $x_0 = 4$ and the closest (in fact, the only) singular point of the coefficient functions $p(x)$ and $q(x)$ is $x = 1$. Since

  \[ \|4 - 1\| = 3 \]

  we can conclude that the radius of convergence of a series solution of the form

  \[ y(x) = \sum_{n=0}^{\infty} a_n (x - 4)^n \]

  will be at least 3. In other words, the series solution will be valid for all $x$ in the interval

  \[ |x - 4| < 3 \]

  or, equivalently, for all $x$ such that

  \[ 1 < x < 7 \]

(c) $(4 + x^2)y'' + 4xy' + y = 0$, $x_0 = 0$

- In this case, the coefficient functions
  
  \[ p(x) = \frac{4x}{4 + x^2} \]
  
  \[ q(x) = \frac{1}{4 + x^2} \]

  both have singularities when

  \[ 4 + x^2 = 0 \Rightarrow x = \pm 2i \]
These two singularities correspond to the points \((0, \pm 2)\) when we represent points in the complex plane as points in the two dimensional plane. Under this representation of the complex plane, the expansion point \(x_0 = 0\) corresponds to the point \((0, 0)\). Therefore the distances between the expansion point and the singularity are

\[
\text{dist}(2i, 0) = \sqrt{(0 - 0)^2 + (2 - 0)^2} = 2 \\
\text{dist}(-2i, 0) = \sqrt{(0 - 0)^2 + (-2 - 0)^2} = 2
\]

Hence, the minimal distance is 2, and so the radius of convergence of a power series solution about 0 is at least 2.

(d) \((1 + x^2)y'' + 4xy' + y = 0\), \(x_0 = 2\)

- In this case the coefficient functions
  \[
  p(x) = \frac{4x}{1 + x^2} \\
  q(x) = \frac{1}{1 + x^2}
  \]

both have singularities when

\[
1 + x^2 = 0 \implies x = \pm i
\]

These two singularities correspond to the points \((0, \pm 1)\) when we represent points in the complex plane as points in the two dimensional plane. Under this representation of the complex plane, the expansion point \(x_0 = 2\) corresponds to the point \((2, 0)\). Therefore the distances between the expansion point and the singularity are

\[
\text{dist}(i, 2) = \sqrt{(0 - 2)^2 + (1 - 0)^2} = \sqrt{5} \\
\text{dist}(-i, 2) = \sqrt{(0 - 2)^2 + (-1 - 0)^2} = \sqrt{5}
\]

Hence, the distance between the expansion point and the closest singularity is \(\sqrt{5}\) and so the radius of convergence of a power series solution about the point \(x_0 = 2\) will be at least \(\sqrt{5}\).

2. Determine the singular points of the following differential equations and state whether they are regular or irregular singular points.

(a) \(xy'' + (1 - x)y' + xy = 0\)

- In this case, the coefficient functions are
  \[
  p(x) = \frac{1 - x}{x} \\
  q(x) = 1
  \]

Since \(p(x)\) is undefined for \(x = 0\), 0 is a singular point. Since the limits

\[
\lim_{x \to 0} (x - 0)p(x) = \lim_{x \to 0}(1 - x) = 1 \\
\lim_{x \to 0} (x - 0)^2q(x) = \lim_{x \to 0}x = 0
\]

both exist, \(x = 0\) is a regular singular point. Alternatively, one could say that because the degree of the singularity of the function \(p(x)\) at the point \(x = 0\) is less than or equal to 1 and the degree of the singularity of the function \(q(x)\) is less than or equal to 2, we have regular singular point at \(x = 0\).

(b) \(x^2(1 - x)^2y'' + 2xy + 4y = 0\)

- In this case, the coefficient functions are
  \[
  p(x) = \frac{2}{x(1 - x)^2} \\
  q(x) = \frac{4}{x^2(1 - x)^2}
  \]
3. The following differential equation has a regular singular point at $x = 0$. Determine the indicial equations, the roots of the indicial equations, the recursion relations, and the first four terms of two linearly independent series solutions.

$$2xy'' + y' + xy = 0$$

- We make the ansatz

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r} , \quad a_0 \neq 0$$

and plug into the differential equation to obtain

$$0 = 2x \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} + \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$+ x \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$= \sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$+ \sum_{n=0}^{\infty} a_n x^{n+r+1}$$

$$= \sum_{n=0}^{\infty} \left[ (2(n+r)(n+r-1) + (n+r)) a_n x^{n+r-1} + a_{n-2} x^{n+r+1} \right]$$

$$= (2r(r-1) + r) a_0 x^{r-1} + (2(1+r)r + 1 + r) a_1 x^r$$

$$+ \sum_{n=1}^{\infty} \left[ (2(n+r)(n+r-1) + (n+r)) a_n - a_{n-2} \right] x^{n+r-1}$$

$$= (2r^2 - r) a_0 x^{r-1} + (2r^2 + 3r + 1) a_1 x^r$$

$$+ \sum_{n=1}^{\infty} \left[ (2(n+r)^2 - (n+r)) a_n + a_{n-2} \right] x^{n+r-1}$$

The function $p(x)$ evidently has a singularity of degree 1 at $x = 0$ and a singularity of degree 2 at $x = 1$. The function $q(x)$ has singularities of degree 2 at $x = 0$ and $x = 1$. In order to be a regular singular point, the degree of the singularity of $p(x)$ must not exceed 1 and the degree of the singularity of $q(x)$ must not exceed 2. Therefore, $x = 0$ is a regular singular point and $x = 1$ is an irregular singular point.

(c) $(1 - x^2)^2 y'' + x(1 - x)y' + (1 + x)y = 0$

- In this case, the coefficient functions are

$$p(x) = \frac{x(1-x)}{(1-x)^2} = \frac{x}{(1-x)^2(1+x)^2}$$

$$q(x) = \frac{4}{x^2(1-x)^2} = \frac{1}{(1-x)^2(1+x)^2}$$

The function $p(x)$ evidently has a singularity of degree 1 at $x = 1$ and a singularity of degree 2 at $x = -1$. The function $q(x)$ has a singularity of degree 1 at $x = 1$ and a singularity of degree 2 at $x = -1$. In order to be a regular singular point the degree of the singularity of $p(x)$ must not exceed 1 and the degree of the singularity of $q(x)$ must not exceed 2. Therefore, $x = 1$ is a regular singular point and $x = -1$ is an irregular singular point.
Setting the total coefficient of \(x^{r-1}, x^r,\) and \(x^{n+r-1}\) equal to zero we obtain the following equations

\[
0 = (2r - 1) a_0 \\
0 = (2r + 1)(r + 1) a_1 \\
a_n = \frac{-a_{n-2}}{2(n + r)^2 - (n + r)}
\]

Since \(a_0\) is assumed to be non-zero the first equation leads to

\[
0 = r(2r - 1) \implies r = 0, 1
\]

If \(r = 0\), then the second equation produces

\[
0 = (0 + 1)(0 + 1)a_1 = a_1 \implies a_1 = 0
\]

The third equation furnishes recursion relations that allow us to express all the even coefficients \(a_{2i}\) in terms of \(a_0\) and all the odd coefficients in terms of \(a_1\). However, because \(a_1 = 0\) only even powers of \(x\) will occur.

To see this, let us first take \(r = 0\). Then the recursion relation is

\[
a_n = \frac{-a_{n-2}}{2n^2 - n}
\]

so

\[
a_2 = \frac{a_0}{8 - 2} = \frac{-a_0}{6} \\
a_3 = \frac{-a_1}{18 - 3} = 0 \\
a_4 = \frac{-a_2}{36 - 4} = \frac{a_0}{192} \\
a_5 = \frac{-a_3}{50 - 5} = 0
\]

\[\vdots\]

Thus to order \(x^5\) one solution will be

\[
y_1(x) = a_0 \left( 1 - \frac{1}{6}x^2 + \frac{1}{192}x^4 + \cdots \right)
\]

To get a second linearly independent solution we solve the recursion relations when \(r = \frac{1}{2}\):

\[
a_n = \frac{-a_{n-2}}{2(n + \frac{1}{2})^2 + (n + \frac{1}{2})} = \frac{-2a_{n-2}}{(2n + 1)^2 - (2n + 1)}
\]

\[
a_2 = \frac{-2a_0}{25 - 5} = \frac{1}{10}a_0 \\
a_3 = \frac{-2a_1}{49 - 7} = 0 \\
a_4 = \frac{-2a_2}{81 - 9} = \frac{1}{300}a_0 \\
a_5 = \frac{-2a_3}{121 - 11} = 0
\]

\[\vdots\]

So we also have a solution (up to order \(x^5\))

\[
y_2 = a_0 \left( 1 - \frac{1}{10}x^2 + \frac{1}{300}x^4 + \cdots \right)
\]
The following differential equations have a regular singular point at \( x = 0 \). Determine the indicial equation and the recursion relations corresponding to the largest root of the indicial equation. Write down the first four terms of the corresponding series expansion.

(a) \( xy'' + y = 0 \)

- This differential equation has a regular singular point at \( x = 0 \). Setting
  \[
y(x) = \sum_{n=0} a_n x^{n+r}, \quad a_0 \neq 0 ,
\]

and plugging into the differential equation we get
  \[
  0 = x \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} + \sum_{n=0}^{\infty} a_n x^{n+r}
  \]

In order to combine the two series on the right we first shift the summation on the first by
  \[
k + r = n + r - 1 \quad \Rightarrow \quad k = n - 1 \quad n = k + 1
\]
to obtain
  \[
  0 = \sum_{k=-1}^{\infty} (k+1+r)(k+r)a_{k+1} x^{k+r} + \sum_{k=0}^{\infty} a_k x^{k+r}
  \]

Demanding that the total coefficient of each power of \( x \) vanish we thus obtain
  \[
  0 = r(r-1)a_0 \quad \Rightarrow \quad r = 0, 1
  \]

Noting that the two roots of the indicial equation \( r(r-1) = 0 \) differ only by an integer, we follow the instructions in the statement of the problem and look for a solution corresponding to the larger root \( r = 1 \).

For this value of \( r \) the recursion relations are
  \[
a_{k+1} = -\frac{-a_k}{(k+2)(k+1)}
\]

Thus,
  \[
a_1 = \frac{-a_0}{2}, \quad a_2 = -\frac{a_0}{2}, \quad a_3 = -\frac{a_0}{144}
\]
Thus the first four terms of this series solution will be

\[ y(x) = \sum_{n=0}^{\infty} a_n x^{n+1} \]

\[ = a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + \cdots \]

\[ = a_0 x - \frac{a_0}{2} x^2 + \frac{a_0}{12} x^3 - \frac{a_0}{144} x^4 \]

\[ = a_0 \left( x - \frac{1}{2} x^2 + \frac{1}{12} x^3 - \frac{1}{144} x^4 + \cdots \right) \]

(b) \( xy'' + (1 - x)y' - y = 0 \)

- Setting

\[ y(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \]

and plugging in we obtain

\[ 0 = x \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} + (1 - x) \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} \]

\[ - \sum_{n=0}^{\infty} a_n x^{n+r} \]

\[ = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} \]

\[ - \sum_{n=0}^{\infty} (n+r)a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} \]

\[ = \sum_{k=0}^{\infty} (k+1+r)(k+r)a_{k+1} x^{k+r} + \sum_{k=0}^{\infty} (k+1+r)a_{k+1} x^{k+r} \]

\[ - \sum_{k=0}^{\infty} (k+r)a_k x^{k+r} - \sum_{k=0}^{\infty} a_k x^{k+r} \]

\[ = (r+1)a_0 x^{r-1} + \sum_{k=0}^{\infty} (k+1+r)(k+r)a_{k+1} x^{k+r} \]

\[ + ra_0 x^{r-1} + \sum_{k=0}^{\infty} (k+1+r)a_{k+1} x^{k+r} \]

\[ - \sum_{k=0}^{\infty} (k+r)a_k x^{k+r} - \sum_{k=0}^{\infty} a_k x^{k+r} \]

\[ = r^2 a_0 x^{r-1} \]

\[ + \sum_{k=0}^{\infty} ((k+r+1)(k+r) + (k+r+1)) a_{k+1} - ((k+r) + 1) a_k \] \( x^{k+r} \)

Setting the total coefficient of each power of \( x \) equal to zero we obtain

\[ a_{k+1} = \frac{a_k}{k+r+1} \]

\[ r^2 = 0 \]

\[ a_k = \frac{r^2}{k+r+1} \]
The indicial equation $r^2 = 0$ implies $r = 0$, and so the recursion relations become

\[ a_{k+1} = \frac{a_k}{k+1}. \]

Hence

\[

da_1 = \frac{a_0}{1} = a_0 \\
da_2 = \frac{a_1}{2} = \frac{1}{2}a_0 \\
da_3 = \frac{a_2}{3} = \frac{1}{3(2)}a_0 \\
da_4 = \frac{a_3}{4} = \frac{1}{4(3)(2)}a_0 \\
& \quad \vdots \\
a_n = \frac{1}{n!}a_0.
\]

Thus,

\[
y(x) = \sum_{n=0}^{\infty} a_n x^{n+0} \\
&= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
&= e^x.
\]