Math 2233
Homework Set 7

1. Find the general solution to the following differential equations. If initial conditions are specified, also determine the solution satisfying those initial conditions.

(a) $y^{(4)} + 2y'' + y = 0$

- The characteristic equation is

$$0 = \lambda^4 + 2\lambda^2 + y = (\lambda^2 + 1)^2 = ((\lambda - i)(\lambda + i))^2 = (\lambda - i)^2 (\lambda + i)^2$$

We thus have two complex roots, $\lambda = +i, -i$ each with multiplicity two. The corresponding linearly independent solutions are

$$\begin{aligned} y_1(x) &= \cos(x) \\ y_2(x) &= x\cos(x) \\ y_3(x) &= \sin(x) \\ y_4(x) &= x\sin(x) \end{aligned}$$

and the general solution is

$$y(x) = c_1 \cos(x) + c_2 x \cos(x) + c_3 \sin(x) + c_4 x \sin(x)$$

(b) $y''' - y'' - y' + y = 0$

- The characteristic equation is

$$0 = \lambda^3 - \lambda^2 - \lambda + 1$$

Note that the right hand side vanishes when $\lambda = 1$; therefore $(\lambda - 1)$ must be a factor of $\lambda^3 - \lambda^2 - \lambda + 1$. Indeed,

$$(\lambda - 1)\,\overline{|\,\lambda^3 - \lambda^2 - \lambda + 1} = \lambda^2 - 1 = (x - 1)(x + 1)$$

So the characteristic polynomial factors as

$$\lambda^3 - \lambda^2 - \lambda + 1 = (\lambda - 1)^2(\lambda + 1)$$

Thus we have a double root at $\lambda = 1$ and a single root at $\lambda = -1$. The corresponding linearly independent solutions are

$$\begin{aligned} y_1(x) &= e^x \\ y_2(x) &= xe^x \\ y_3(x) &= e^{-x} \end{aligned}$$

and the general solution is

$$y(x) = c_1 e^x + c_2 x e^x + c_3 e^{-x}$$

(c) $y''' - 3y'' + 3y' - y = 0$, $y(0) = 1$, $y'(0) = 2$, $y''(0) = 3$

- The characteristic equation is

$$0 = \lambda^3 - 3\lambda^2 + 3\lambda - 1$$

Again $\lambda = 1$ is an obvious solution and so $(\lambda - 1)$ is a factor of $\lambda^3 - 3\lambda^2 + 3\lambda - 1$. To find the remaining factors we employ polynomial division and find

$$(\lambda - 1)\,\overline{|\,\lambda^3 - 3\lambda^2 + 3\lambda - 1} = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$$

and so

$$0 = \lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)(\lambda - 1)^2 = (\lambda - 1)^3$$

We thus have a triple root $\lambda = 1$. The corresponding linearly independent solutions are

$$\begin{aligned} y_1(x) &= e^x \\ y_2(x) &= xe^x \\ y_3(x) &= x^2e^x \end{aligned}$$

and the general solution is

$$y(x) = c_1e^x + c_2xe^x + c_3x^2e^x$$

We shall now impose the initial condtions to fix the arbitrary constants c_1, c_2, and c_3.

$$\begin{aligned} 1 &= y(0) = c_1e^0 + c_2(0)e^0 + c_3(0)^2e^0 = c_1 \\ 2 &= y'(0) = c_1e^0 + c_2\left(e^0 + (0)e^0\right) + c_3\left(2(0)e^0 + (0)^2e^0\right) = c_1 + c_2 \\ 3 &= y''(0) = c_1e^0 + c_2\left(e^0 + e^0 + (0)e^0\right) + c_3 2e^o + 2(0)e^0 + 2(0)e^0 + (0)^2e^0 = c_1 + 2c_2 + 2c_3 \end{aligned}$$

and so we have

$$\begin{aligned} c_1 &= 1 \\ c_2 &= 2 - c_1 = 2 - 1 = 1 \\ c_3 &= \frac{1}{2}(3 - c_1 - 2c_2) = \frac{1}{2}(3 - 1 - 2) = 0 \end{aligned}$$

The solution to the initial value problem is thus

$$y(x) = e^x + xe^x$$

(d) $y''' + 5y'' - y' - 5y = 0$

- The characteristic equation is

$$0 = \lambda^3 + 5\lambda^2 - \lambda + 5$$

Again we are lucky enough to spot the solution $\lambda = 1$ and so we can identify the other roots by factoring the right hand side of

$$(\lambda - 1)\,\overline{\left|\,\lambda^3 + 5\lambda^2 - \lambda + 5\right.} = \lambda^2 - 4\lambda + 5$$

Obviously, $\lambda^2 - 4\lambda + 5 = (\lambda - 5)(\lambda + 1)$, and so the right hand side of the characteristic equation factors as

$$0 = (\lambda - 1)(\lambda - 5)(\lambda + 1)$$

We thus have three distinct roots $\lambda = 1, 5, -1$. The corresponding linearly independent solutions are

$$\begin{aligned} y_1(x) &= e^x \\ y_2(x) &= e^{5x} \\ y_3(x) &= e^{-x} \end{aligned}$$

and the general solution is

$$y(x) = c_1e^x + c_2e^{5x} + c_3e^{-x}$$

(e) $y^{(4)} - 9y'' = 0$

The characteristic equation is

$$0 = \lambda^4 - 9\lambda^2 = \lambda^2(\lambda^2 - 9) = \lambda^2(\lambda - 3)(\lambda + 3) = (\lambda - 0)^2(\lambda - 3)(+3)$$

We thus have a double root at $\lambda = 0$, and single roots at $\lambda = \pm 3$. The corresponding linearly independent solutions are

$$\begin{aligned} y_1(x) &= e^{0x} = 1 \\ y_2(x) &= xe^{0x} = x \\ y_3(x) &= e^{3x} \\ y_4(x) &= e^{-3x} \end{aligned}$$

and the general solution is

$$y(x) = c_1 + c_2 x + c_3 e^{3x} + c_4 e^{-3x}$$

2. Combine each of the following power series expressions into a single power series.

(a) $\displaystyle\sum_{n=1}^{\infty}(n+1)(x-1)^{n-1} + \sum_{n=0}^{\infty} n(x-1)^n$

- First we shift the summation index of the first power series up by 1 (remembering to shift its starting point down by 1);

$$\sum_{n=1}^{\infty}(n+1)(x-1)^{n-1} + \sum_{n=0}^{\infty} n(x-1)^n = \sum_{n=0}^{\infty}(n+2)(x-1)^n + \sum_{n=0}^{\infty} n(x-1)^n$$

Now both power series start off at the same value of n and involve the same powers of $(x-1)$; so we can combine them into a single power series by simply adding the coefficients of like powers of $(x-1)$. Hence,

$$\begin{aligned} \sum_{n=1}^{\infty}(n+1)(x-1)^{n-1} + \sum_{n=0}^{\infty} nx^n &= \sum_{n=0}^{\infty}\left((n+2)+n\right)(x-1)^n \\ &= \sum_{n=0}^{\infty}(2n+2)(x-1)^n \end{aligned}$$

(b) $\displaystyle\sum_{n=0}^{\infty}(n+1)a_{n+2}x^{n+1} + \sum_{n=0}^{\infty} na_n x^{n-1}$

- This problem can be handled several different ways. We could shift the summation index of the first series down by 2, or we could shift the summation index of the second series up by 2, or we could shift the first up by 1 and the second down by 1. Let's do it the first way

$$\begin{aligned} \sum_{n=0}^{\infty}(n+1)a_{n+2}x^{n+1} + \sum_{n=0}^{\infty} na_n x^{n-1} &= \sum_{n=2}^{\infty}(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} na_{n+2}x^{n-1} \\ &= \sum_{n=2}^{\infty}(n-1)a_n x^{n-1} + 0a_2x^{-1} + 1a_3x^0 + \sum_{n=2}^{\infty} na_{n+2}x^{n-1} \\ &= a_3 + \sum_{n=2}^{\infty}\left((n-1)a_n + na_{n+2}\right)x^{n-1} \\ &= a_3 + \sum_{n=1}^{\infty}\left((n-1)a_{n+1} + (n+1)a_{n+3}\right)x^n \end{aligned}$$

In the last step we shifted the summation index again just so that it would be easy to identify the total coefficient of the n^{th} power of x in terms of n.

(c) $\displaystyle(x-1)\sum_{n=0}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n$

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$$\begin{aligned}
(x-1)\sum_{n=0}^{\infty}{}_{n}na_nx^{n-1}+\sum_{n=0}^{\infty}a_nx^n &= x\sum_{n=0}^{\infty}na_nx^{n-1}-\sum_{n=0}^{\infty}na_nx^{n-1}+\sum_{n=0}^{\infty}a_nx^n \\
&= \sum_{n=0}^{\infty}na_nx^n-\sum_{n=0}^{\infty}na_nx^{n-1}+\sum_{n=0}^{\infty}a_nx^n \\
&= \sum_{n=0}^{\infty}na_nx^n-\sum_{n=-1}^{\infty}(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}a_nx^n \\
&= \sum_{n=0}^{\infty}na_nx^n-\left((-1+1)a_{-1+1}x^{-1}+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n\right) \\
&\quad +\sum_{n=0}^{\infty}a_nx^n \\
&= \sum_{n=0}^{\infty}na_nx^n-\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}a_nx^n \\
&= \sum_{n=0}^{\infty}\left(na_n-(n+1)a_{n+1}+a_n\right)x^n \\
&= \sum_{n=0}^{\infty}(n+1)\left(a_n-a_{n+1}\right)x^n
\end{aligned}$$

(d) $x\sum_{n=0}^{\infty}na_n(x-1)^{n-1}+\sum_{n=0}^{\infty}a_n(x-1)^n$

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$$\begin{aligned}
x\sum_{n=0}^{\infty}na_n(x-1)^{n-1}+\sum_{n=0}^{\infty}a_n(x-1)^n &= x\sum_{n=0}^{\infty}na_n(x-1)^{n-1}+\sum_{n=0}^{\infty}a_n(x-1)^n \\
&= [(x-1)+1]\sum_{n=0}^{\infty}na_n(x-1)^{n-1}+\sum_{n=0}^{\infty}a_n(x-1)^n \\
&= \sum_{n=0}^{\infty}na_n(x-1)^n+\sum_{n=0}^{\infty}na_n(x-1)^{n-1}+\sum_{n=0}^{\infty}a_n(x-1)^n \\
&= \sum_{n=0}^{\infty}na_n(x-1)^n+\sum_{n=-1}^{\infty}(n+1)a_{n+1}(x-1)^n+\sum_{n=0}^{\infty}a_n(x-1)^n \\
&= \sum_{n=0}^{\infty}na_n(x-1)^n+\left((0)a_0(x-1)^0+\sum_{n=-1}^{\infty}(n+1)a_{n+1}(x-1)^n\right) \\
&\quad +\sum_{n=0}^{\infty}a_n(x-1)^n \\
&= \sum_{n=0}^{\infty}na_n(x-1)^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^n+\sum_{n=0}^{\infty}a_n(x-1)^n \\
&= \sum_{n=0}^{\infty}\left(na_n+(n+1)a_{n+1}+a_n\right)(x-1)^n \\
&= \sum_{n=0}^{\infty}\left((n+1)a_{n+1}+(n+1)a_n\right)(x-1)^n
\end{aligned}$$

□

(e) $x^2 \sum_{n=0}^{\infty} n(n-1)a_n (x-1)^{n-2}$

- We shall begin by calculating the Taylor expansion of x^2 about $x_o = 1$.

$$\begin{aligned}
x^2 &= f(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \cdots \\
&= 1 + 2(x-1) + \frac{2}{2!}(x-1)^2 + \frac{0}{3!}(x-1)^3 + \frac{0}{4!}(x-1)^4 + \cdots \\
&= 1 + 2(x-1) + (x-1)^2
\end{aligned}$$

Therefore

$$\begin{aligned}
x^2 \sum_{n=0}^{\infty} n(n-1)a_n(x-1)^{n-2} &= \left(1 + 2(x-1) + (x-1)^2\right) \sum_{n=0}^{\infty} n(n-1)a_n(x-1)^{n-2} \\
&= \sum_{n=0}^{\infty} n(n-1)a_n(x-1)^{n-2} + 2(x-1)\sum_{n=0}^{\infty} n(n-1)a_n(x-1)^{n-2} \\
&\quad +(x-1)^2 \sum_{n=0}^{\infty} n(n-1)a_n(x-1)^{n-2} \\
&= \sum_{n=0}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=0}^{\infty} 2n(n-1)a_n(x-1)^{n-1} \\
&\quad + \sum_{n=0}^{\infty} n(n-1)a_n(x-1)^{n} \\
&= \sum_{n=-2}^{\infty} (n+2)(n+1)a_{n+2}(x-1)^n + \sum_{n=-1}^{\infty} 2(n+1)(n)a_{n+1}(x-1)^n \\
&\quad + \sum_{n=0}^{\infty} n(n-1)a_n(x-1)^{n} \\
&= \left(0 + 0 + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n\right) \\
&\quad + \left(0 + \sum_{n=0}^{\infty} 2(n+1)(n)a_{n+1}(x-1)^n\right) \\
&\quad + \sum_{n=0}^{\infty} n(n-1)a_n(x-1)^{n} \\
&= \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + 2n(n+1)a_{n+1} + n(n-1)a_n\right](x-1)^n
\end{aligned}$$

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$\square$

3. Find the recursion relations for the power series solutions $y(x) = \sum_{n=0}^{\infty} a_n(x-x_o)^n$ of the following differential equations

(a) $y'' - xy' - y$, $x_o = 0$

- Setting

$$y(x) = \sum_{n=0}^{\infty} a_n\,(x-0)^n = \sum_{n=0}^{\infty} a_n x^n$$

into the differential equation (**??**), we get

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=0}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=0}^{\infty} n a_n x^{n} - \sum_{n=0}^{\infty} a_n x^n \quad . \end{aligned}$$

We now shift the first series by 2 by setting

$$\begin{aligned} k &= n-2 \\ n &= k+2 \end{aligned}$$

and simply relabel the counting index of the last two series by k.

$$\begin{aligned} 0 &= \sum_{k=-2}^{\infty} (k+2)(k+1)a_{k+2}x^k - \sum_{k=0}^{\infty} k a_k x^k \sum_{k=0}^{\infty} a_k x^k \\ &= (-2+2)(-2+1)a_0 x^{-2} + (-1+2)(-1+1)a_1 x^{-1} + \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k \\ &\quad - \sum_{k=0}^{\infty} k a_k x^k - \sum_{k=0}^{\infty} a_k x^k \\ &= 0 + 0 + \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k - \sum_{k=0}^{\infty} k a_k x^k - \sum_{k=0}^{\infty} a_k x^k \\ &= \sum_{k=0}^{\infty} \left((k+2)(k+1)a_{k+2} - k a_{k-} a_k\right) x^k \end{aligned}$$

In order to ensure that the right hand side vanish for all x we now demand the total coefficient of x^k vanish for all k. Thus

$$(k+2)(k+1)a_{k+2} - (k+1)a_k = 0 \qquad , \qquad \text{for all } k \quad ,$$

or, after solving for a_{k+2} and then replacing k by n

$$a_{n+2} = \frac{a_n}{(n+2)} \quad .$$

The above equation is the recursion relation for the coefficients a_k. □

(b) $y'' - xy' - y = 0$, $x_o = 1$

- We set

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$$

and plug into the differential equation:

$$0 = \sum_{n=2}^{\infty} n(n-1)a_n (x-1)^{n-2} - x\sum_{n=1}^{\infty} n a_n (x-1)^{n-1} - \sum_{n=0}^{\infty} a_n (x-1)^n \quad .$$

The hardest part about this problem will be to combine the three power series appearing on the right hand side of the equation above into a single power series. We have

$$\begin{aligned}
0 &= \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - x\sum_{n=1}^{\infty} na_n(x-1)^{n-1} - \sum_{n=0}^{\infty} a_n(x-1)^n \\
&= \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - (x-1+1)\sum_{n=2}^{\infty} na_n(x-1)^{n-1} - \sum_{n=0}^{\infty} a_n(x-1)^n \\
&= \sum_{n=1}^{\infty} n(n-1)a_n(x-1)^{n-2} - (x-1)\sum_{n=1}^{\infty} na_n(x-1)^{n-1} - \sum_{n=1} na_n(x-1)^{n-1} \\
&\qquad - \sum_{n=0}^{\infty} a_n(x-1)^n \\
&= \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - \sum_{n=1}^{\infty} na_n(x-1)^n - \sum_{n=1}^{\infty} na_n(x-1)^{n-1} \\
&\qquad - \sum_{n=0}^{\infty} a_n(x-x_o)^n \\
&= \sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}(x-1)^m - \sum_{n=0}^{\infty} na_n(x-1)^n \\
&\qquad - \sum_{k=0}^{\infty}(k+1)a_{k+1}(x-1)^k - \sum_{n=0}^{\infty} a_n(x-1)^n \\
&= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n - \sum_{n=0} na_n(x-1)^n \\
&\qquad - \sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^n - \sum_{n=0}^{\infty} a_n(x-1)^n \\
&= \sum_{n=0}^{\infty}\left((n+2)(n+1)a_{n+2} - na_n - (n+1)a_{n+1} + a_n\right)(x-1)^n
\end{aligned}$$

We must therefore have

$$0 = (n+2)(n+1)a_{n+2} - na_n - (n+1)a_{n+1} - a_n$$

or

$$a_{n+2} = \frac{a_n + a_{n+1}}{(n+2)} \quad .$$

□

(c) $(1-x)y'' + y = 0$, $x_o = 0$

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$$(1-x)y'' + y = 0 \qquad , \qquad x_o = 0$$

Since $x_o = 0$, we set

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

and plug this expression for $y(x)$ into the differential equation. This yields

$$\begin{aligned}
0 &= (1-x)\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n
\end{aligned}$$

We now perform the usual shifts of summation indices (in order to put each series in standard form) and then combine the above sum of three series into a single series.

$$\begin{aligned}
0 &= \sum_{k=-2}^{\infty}(k+2)(k+1)a_{k+2}x^k - \sum_{k=-1}^{\infty}(k+1)(k)a_{k+1}x^k + \sum_{k=0}^{\infty}a_kx^k \\
&= 0+0+\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k - 0 - \sum_{k=0}^{\infty}(k+1)(k)a_{k+1}x^k + \sum_{k=0}^{\infty}a_kx^k \\
&= \sum_{k=0}^{\infty}\left((k+2)(k+1)a_{k+2} - k(k+1)a_{k+1} + a_k\right)x^k
\end{aligned}$$

We can now read off the recursion relation:

$$(k+2)(k+1)a_{k+2} - k(k+1)a_{k+1} + a_k = 0$$

or, after solving for a_{k+2} and then replacing k by n

$$a_{n+2} = \frac{n(n+1)a_{n+1} - a_n}{(n+2)(n+1)}$$

□

(d) $y'' + xy' + 2y = 0$, $x_o = 0$

- Since $x_o = 0$, we set

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

and plug this expression for $y(x)$ into the differential equation. This yields

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty} n(n-1)a_nx^{n-2} + x\sum_{n=0}^{\infty} na_nx^{n-1} + 2\sum_{n=0}^{\infty}a_nx^n \\
&= \sum_{n=0}^{\infty} n(n-1)a_nx^{n-2} + \sum_{n=0}^{\infty} na_nx^{n} + \sum_{n=0}^{\infty}2a_nx^n
\end{aligned}$$

We now perform the usual shifts of summation indices (in order to put each series in standard form) and then combine the above sum of three series into a single series.

$$\begin{aligned}
0 &= \sum_{k=-2}^{\infty}(k+2)(k+1)a_{k+2}x^k + \sum_{k=0}^{\infty}ka_kx^k + \sum_{k=0}^{\infty}2a_kx^k \\
&= 0+0+\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k + \sum_{k=0}^{\infty}ka_kx^k + \sum_{k=0}^{\infty}2a_kx^k \\
&= \sum_{k=0}^{\infty}\left((k+2)(k+1)a_{k+2} + (k+2)a_k\right)x^k
\end{aligned}$$

Our recursion relations are thus,

$$(k+2)(k+1)a_{k+2} + (k+2)a_k = 0$$

or, after solving for a_{k+2} and then replacing k by n

$$a_{n+2} = \frac{a_n}{n+1} \quad .$$

□

(e) $(1+x^2)y'' - 4xy' + 6y = 0$, $x_o = 0$

- Since $x_o = 0$, we set

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

and plug this expression for $y(x)$ into the differential equation. This yields

$$\begin{aligned} 0 &= (1+x^2)\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - 4x\sum_{n=0}^{\infty} n a_n x^n + 6\sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 6a_n x^n \end{aligned}$$

We now perform the usual shifts of summation indices (in order to put each series in standard form) and then combine the above sum of three series into a single series.

$$\begin{aligned} 0 &= \sum_{k=-2}^{\infty} (k+2)(k+1)a_{k+2}x^k + \sum_{k=0}^{\infty} k(k-1)a_k x^k - \sum_{k=0}^{\infty} 4k a_k x^k + \sum_{k=0}^{\infty} 6a_k x^k \\ &= 0 + 0 + \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k + \sum_{k=0}^{\infty} k(k-1)a_k x^k - \sum_{k=0}^{\infty} 4k a_k x^k + \sum_{k=0}^{\infty} 6a_k x^k \\ &= \sum_{k=0}^{\infty} \left((k+2)(k+1)a_{k+2} + \left(k^2 - 5k + 6\right) a_k\right) x^k \\ &= \sum_{k=0}^{\infty} \left((k+2)(k+1)a_{k+2} + (k-2)(k-3)a_k\right) x^k \end{aligned}$$

We thus arrive at the following recursion relation $(k+2)(k+1)a_{k+2} + (k-2)(k-3)a_k = 0$. After solving for a_{k+2} and then replacing k by n

$$a_{n+2} = \frac{-(n-2)(n-3)a_n}{(n+2)(n+1)} \quad .$$

$\square$

4. Find power series expressions for the general solutions of the following differential equations. (You may utilize recursion relations found in Problem 3.)

(a) $y'' - xy' - y = 0$, $x_o = 0$

- In Problem 3a we found that the recursion relations for a power series solution about $x_o = 0$ for this differential equation are

$$a_{n+2} = \frac{a_n}{n+2}$$

We will now apply these recursion relations for $n = 0, 1, 2, 3, \ldots$ to get expressions for the coefficients $a_2, a_3, a_4, a_5, \ldots$ in terms of the first two coefficients a_0 and a_1.

$$\begin{aligned} n = 0 \quad &\Rightarrow \quad a_2 = a_{0+2} = \tfrac{a_0}{0+2} = \tfrac{a_0}{2} \\ n = 1 \quad &\Rightarrow \quad a_3 = a_{1+2} = \tfrac{a_1}{1+2} = \tfrac{a_1}{3} \\ n = 2 \quad &\Rightarrow \quad a_4 = a_{2+2} = \tfrac{a_2}{2+2} = \tfrac{a_2}{4} = \tfrac{a_0}{2\cdot 4} \\ n = 3 \quad &\Rightarrow \quad a_5 = a_{3+2} = \tfrac{a_3}{3+2} = \tfrac{a_3}{5} = \tfrac{a_1}{3\cdot 5} \\ n = 4 \quad &\Rightarrow \quad a_6 = a_{4+2} = \tfrac{a_4}{4+2} = \tfrac{a_4}{6} = \tfrac{a_0}{2\cdot 4\cdot 6} \end{aligned}$$

Hence

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots \\
&= a_0 + a_1 x + \frac{a_0}{2} x^2 + \frac{a_1}{3} x^3 + \frac{a_0}{2\cdot 4} x^4 + \frac{a_1}{3\cdot 5} x^5 + \frac{a_0}{2\cdot 4\cdot 6} x^6 + \cdots \\
&= a_0 \left(1 + \frac{1}{2} x^2 + \frac{1}{2\cdot 4} x^4 + \frac{1}{2\cdot 4\cdot 6} x^6 + \cdots\right) \\
&\quad + a_1 \left(x + \frac{1}{3} x^3 + \frac{1}{3\cdot 5} x^5 + \cdots\right)
\end{aligned}$$

□

(b) $y'' - xy - y = 0$, $x_o = 1$

- n Problem 3b we found that the recursion relations for a power series solution about $x_o = 0$ for this differential equation are

$$a_{n+2} = \frac{a_n + a_{n+1}}{(n+2)}$$

We will now apply these recursion relations for $n = 0, 1, 2, \ldots$ to get expressions for the coefficients $a_2, a_3, a_4, \ldots$ in terms of the first two coefficients a_0 and a_1.

$$\begin{aligned}
n = 0 \quad &\Rightarrow \quad a_2 = a_{0+2} = \tfrac{a_0 + a_1}{0+2} = \tfrac{a_0}{2} + \tfrac{a_1}{2} \\
n = 1 \quad &\Rightarrow \quad a_3 = a_{1+2} = \tfrac{a_1 + a_2}{1+2} = \tfrac{a_1}{3} + \tfrac{1}{3}\left(\tfrac{a_0}{2} + \tfrac{a_1}{2}\right) = \tfrac{a_0}{6} + \tfrac{a_1}{2} \\
n = 2 \quad &\Rightarrow \quad a_4 = a_{2+2} = \tfrac{a_2 + a_3}{2+2} = \tfrac{1}{4}\left(\tfrac{a_0}{2} + \tfrac{a_1}{2}\right) + \tfrac{1}{4}\left(\tfrac{a_0}{6} + \tfrac{a_1}{2}\right) = \tfrac{a_0}{6} + \tfrac{a_1}{4}
\end{aligned}$$

Therefore

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n (x-1)^n \\
&= a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4 + \cdots \\
&= a_0 + a_1(x-1) + \left(\frac{a_0}{2} + \frac{a_1}{2}\right)(x-1)^2 + \left(\frac{a_0}{6} + \frac{a_1}{2}\right)(x-1)^3 + \left(\frac{a_0}{6} + \frac{a_1}{4}\right)(x-1)^4 + \cdots \\
&= a_0 \left(1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{6}(x-1)^4 + \cdots\right) + \\
&\quad a_1 \left((x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{4}(x-1)^4 + \cdots\right)
\end{aligned}$$

□

5. Find power series expressions for the solutions to the following initial value problems. (You may utilize recursion relations found in Problem 3.)

(a) $(1-x)y'' + y = 0$, $y(0) = 2$, $y'(0) = 1$

- In Problem 3c we obtained the following recursion relations for power series solutions of this differential equation (expanded about $x_o = 0$).

$$a_{n+2} = \frac{n(n+1)a_{n+1} - a_n}{(n+2)(n+1)}$$

For such solutions

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

the initial conditions imply

$$\begin{aligned} 2 &= y(0) = a_0 \\ 1 &= y'(0) = a_1 \end{aligned}$$

We now can apply the recursion relations to get numerical values for $a_2, a_3, a_4, \ldots$

$$\begin{aligned} n = 0 &\Rightarrow a_2 = \tfrac{0(0+1)a - + a_0}{2 \cdot 1} = -\tfrac{a_0}{2} = -1 \\ n = 1 &\Rightarrow a_3 = \tfrac{1(1+1)a_2 - a_1}{3 \cdot 2} = \tfrac{2a_2 - a_1}{6} = \tfrac{-2-1}{6} = -\tfrac{1}{2} \\ n = 2 &\Rightarrow a_4 = \tfrac{2(2+1)a_3 - a_2}{4 \cdot 3} = \tfrac{6a - a_2}{12} = \tfrac{-3+1}{12} = -\tfrac{1}{6} \\ n = 3 &\Rightarrow a_5 = \tfrac{3(3+1)a_4 - a_3}{5 \cdot 4} = \tfrac{12a_4 - a_3}{20} = \tfrac{-2+\frac{1}{2}}{20} = -\tfrac{3}{40} \end{aligned}$$

so

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 \\ &= 2 + x - x^2 - \frac{1}{2}x^3 - \frac{1}{6}x^4 - \frac{3}{40}x^5 + \cdots \end{aligned}$$

□

(b) $y'' - xy' - y = 0$, $y(1) = 1$, $y'(1) = 2$

- In Problem 3b we found that the recursion relations for a power series solution about $x_o = 1$ for this differential equation are

$$a_{n+2} = \frac{a_n + a_{n+1}}{(n+2)}$$

Also, also in Problem 4b we found that the power series expansion about $x_0 = 1$ of the general solution to this differential equation is

$$y(x) = a_0 + a_1(x-1) + \left(\frac{a_0}{2} + \frac{a_1}{2}\right)(x-1)^2 + \left(\frac{a_0}{6} + \frac{a_1}{2}\right)(x-1)^3 + \left(\frac{a_0}{6} + \frac{a_1}{4}\right)(x-1)^4 + \cdots$$

All that remains to be done is to use the initial conditions to fix the values of the first two coefficients.

$$\begin{aligned} y(1) = 1 &\Rightarrow a_0 = 1 \\ y'(1) = 2 &\Rightarrow a_1 = 2 \end{aligned}$$

Therefore, the solution to the initial value problem is

$$\begin{aligned} y(x) &= 1 + 2(x-1) + \left(\frac{1}{2} + \frac{2}{2}\right)(x-1)^2 + \left(\frac{1}{6} + \frac{2}{2}\right)(x-1)^3 + \left(\frac{1}{6} + \frac{2}{4}\right)(x-1)^4 + \cdots \\ &= 1 + 2(x-1) + \frac{3}{2}(x-1)^2 + \frac{7}{6}(x-1)^3 + \frac{7}{24}(x-1)^4 + \cdots \end{aligned}$$

□