

Math 2233  
Homework Set 6

1. Solve the following Euler-type equations.

(a)  $x^2y'' + xy' + y = 0$

- Since the differential equation is of Euler-type we expect solutions of the form  $y(x) = x^r$ . Substituting  $y(x) = x^r$  into the differential equation we obtain

$$0 = x^2 (r(r-1)x^{r-2}) + x (rx^{r-1}) + x^r = (r(r-1) + r + 1) x^r = (r^2 + 1) x^r$$

The indicial equation is thus

$$r^2 + 1 = 0$$

This equation has two complex roots  $r = \pm i$ . We thus have the following two linearly independent real-valued solutions

$$\begin{aligned} y_1(x) &= \cos(\ln|x|) \\ y_2(x) &= \sin(\ln|x|) \end{aligned}$$

and the general solution is

$$y(x) = c_1 \cos(\ln|x|) + c_2 \sin(\ln|x|)$$

(b)  $x^2y'' - xy' + 2y = 0$

- Substituting  $y(x) = x^r$  into this equation yields

$$0 = r(r-1)x^r - rx^r + 2x^r = (r^2 - 2r + 2) x^r$$

So we must have

$$r^2 - 2r + 2 = 0$$

or, after applying the quadratic formula,

$$r = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

We thus have two complex roots of the indicial equation and thus the following two linearly independent real-valued solutions of the original Euler-type differential equation

$$\begin{aligned} y_1(x) &= x \cos(\ln|x|) \\ y_2(x) &= x \sin(\ln|x|) \end{aligned}$$

The general solution is thus

$$y(x) = c_1 x \cos(\ln|x|) + c_2 x \sin(\ln|x|)$$

(c)  $4x^2y'' - 4xy' + 3y = 0, y(1) = 0, y'(1) = 1$

- Substituting  $y(x) = x^r$  into this equation yields

$$0 = 4r(r-1)x^r - 4rx^r + 3x^r = (4r^2 - 8r + 3) x^r$$

So we must have

$$4r^2 - 8r + 3 = 0$$

or, after applying the quadratic formula,

$$r = \frac{8 \pm \sqrt{64-48}}{8} = \frac{8 \pm \sqrt{16}}{8} = \frac{8 \pm 4}{8} = 1 \pm \frac{1}{2}$$

We thus have two real roots of the indicial equation;  $r = \frac{3}{2}, \frac{1}{2}$ . The corresponding the linearly independent solutions of the original Euler-type differential equation are

$$\begin{aligned}y_1(x) &= x^{3/2} \\y_2(x) &= x^{1/2}\end{aligned}$$

The general solution is thus

$$y(x) = c_1x^{3/2} + c_2x^{1/2}$$

(d)  $x^2y'' - 3xy' + 3y = 0$

- Substituting  $y(x) = x^r$  into this equation yields

$$0 = r(r-1)x^r - 3rx^r + 3x^r = (r-4r+3)x^r$$

So we must have

$$0 = r^2 - 4r + 3 = (r-1)(r-3)$$

We thus have two real roots of the indicial equation;  $r = 1, 3$ . The corresponding linearly independent real-valued solutions of the original Euler-type differential equation are

$$\begin{aligned}y_1(x) &= x \\y_2(x) &= x^3\end{aligned}$$

The general solution is thus

$$y(x) = c_1x + c_2x^3$$

(e)  $x^2y'' + 5xy' + 5y = 0$

- Substituting  $y(x) = x^r$  into this equation yields

$$0 = r(r-1)x^r + 5rx^r + 5x^r = (r+4r+5)x^r$$

So we must have

$$0 = r^2 + 4r + 5$$

Applying the Quadratic Formula we find

$$r = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm \frac{\sqrt{-4}}{2} = -2 \pm i$$

We thus have a pair of complex roots of the indicial equation;  $r = -2 \pm i$ . The corresponding linearly independent real-valued solutions of the original Euler-type differential equation are

$$\begin{aligned}y_1(x) &= x^{-2} \cos(\ln|x|) \\y_2(x) &= x^{-2} \sin(\ln|x|)\end{aligned}$$

The general solution is thus

$$y(x) = c_1x^{-2} \cos(\ln|x|) + c_2x^{-2} \sin(\ln|x|)$$

2. Given that  $y_1(x) = e^x$  and  $y_2(x) = x$  are solutions of

$$(1-x)y'' + xy' - y = 0$$

find the general solution to

$$(1-x)y'' + xy' - y = 2(x-1)^2e^{-x}$$

by the method of Variation of Parameters.

- In this problem we have

$$g(x) = \frac{2(x-1)^2 e^{-x}}{1-x} = 2(1-x)e^{-x}$$

$$W[y_1, y_2](x) = e^x(x)' - (e^x)'x = e^x(1) - (e^x)x = (1-x)e^x$$

and the Variation of Parameters formula yields

$$\begin{aligned} y_p(x) &= -y_1(x) \int^x \frac{y_2(s)g(s)}{W[y_1, y_2](s)} ds + y_2(x) \int^x \frac{y_1(s)g(s)}{W[y_1, y_2](s)} ds \\ &= -e^x \int^x \frac{s(2(s-1)e^{-s})}{(1-s)e^s} ds + x \int^x \frac{e^s(2(s-1)e^{-s})}{(1-s)e^s} ds \\ &= +2e^x \int^x s e^{-2s} ds - 2x \int^x e^{-s} ds \\ &= 2e^x \left( -\frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2x} \right) - 2x(-e^{-x}) \\ &= -x e^{-x} - \frac{1}{2} e^{-x} + 2x e^{-x} \end{aligned}$$

as a particular solution to the non-homogeneous differential equation. The general solution is thus

$$y(x) = y_p(x) + c_1 y_1(x) + c_2 y_2(x) = -x e^{-x} - \frac{1}{2} e^{-x} + 2x e^{-x} + c_1 e^x + c_2 x$$

3. Find the general solution of each of the following equations by the method of Variation of Parameters. If initial conditions are given, find the solution satisfying that initial value problem.

(a)  $y'' - 3y' + 2y = 10$

- The corresponding homogeneous equation is

(1) 
$$y'' - 3y' + 2y = 0$$

which is second order linear with constant coefficients. Its characteristic equation is

$$\lambda^2 - 3\lambda + 2 = 0$$

or

$$(\lambda - 2)(\lambda - 1) = 0$$

and so we have two real roots  $\lambda = 2, 1$  and two linearly independent solutions of (1):

$$\begin{aligned} y_1(x) &= e^{2x} \\ y_2(x) &= e^x \end{aligned}$$

We will now use the Variation of Parameters to determine a particular solution of the original non-homogeneous equation. First note that

$$\begin{aligned} g(x) &= 10 \\ W[y_1, y_2](x) &= (e^{2x})(e^x) - (2e^{2x})(e^x) = -e^{3x} \end{aligned}$$

and so

$$\begin{aligned}
 y_p(x) &= -y_1(x) \int^x \frac{y_2(s)g(s)}{W[y_1, y_2](s)} ds + y_2(x) \int^x \frac{y_1(s)g(s)}{W[y_1, y_2](s)} ds \\
 &= -e^{2x} \int^x \frac{e^s(10)}{-e^{3s}} ds + e^x \int^x \frac{e^{2s}(10)}{-e^{3s}} ds \\
 &= +10e^{2x} \int^x e^{-2s} ds - 10e^x \int^x e^{-s} ds \\
 &= 10e^{2x} \left( -\frac{1}{2}e^{-2x} \right) - 10e^x (-e^{-x}) \\
 &= -5 + 10 \\
 &= 5
 \end{aligned}$$

Thus the general solution is

$$y(x) = y_p(x) + c_1 y_1(x) + c_2 y_2(x) = 5 + c_1 e^{2x} + c_2 e^x$$

(b)  $y'' + y = \sin(x)$ ,  $y(0) = 1$ ,  $y'(0) = 2$

- The corresponding homogeneous equation is

(2) 
$$y'' + y = 0$$

which is second order linear with constant coefficients. Its characteristic equation is

$$\lambda^2 + 1 = 0$$

or

$$(\lambda + i)(\lambda - i) = 0$$

and so we have two complex roots  $\lambda = \pm 2i$  and two linearly independent solutions of (2):

$$\begin{aligned}
 y_1(x) &= \cos(x) \\
 y_2(x) &= \sin(x)
 \end{aligned}$$

We will now use the Variation of Parameters to determine a particular solution of the original non-homogeneous equation. First note that

$$\begin{aligned}
 g(x) &= 3 \sin(x) \\
 W[y_1, y_2](x) &= (\cos(x))(\cos(x)) - (\sin(x))(\sin(x)) = \cos^2(x) + \sin^2(x) = 1
 \end{aligned}$$

and so

$$\begin{aligned}
 y_p(x) &= -y_1(x) \int^x \frac{y_2(s)g(s)}{W[y_1, y_2](s)} ds + y_2(x) \int^x \frac{y_1(s)g(s)}{W[y_1, y_2](s)} ds \\
 &= -\cos(x) \int^x \frac{\sin(s)(\sin(s))}{1} ds + \sin(x) \int^x \frac{\cos(s)(\sin(s))}{1} ds \\
 &= -\cos(x) \int^x \sin^2(s) ds + \sin(x) \int^x \cos(s) \sin(s) ds \\
 &= -\cos(x) \left( -\frac{1}{2} \cos(x) \sin(x) + \frac{1}{2} x \right) + \frac{1}{2} \sin(x) \left( \frac{1}{2} \sin^2(x) \right) \\
 &= \frac{1}{2} \cos^2(x) \sin(x) - \frac{1}{2} x \cos(x) + \frac{1}{4} \sin^3(x)
 \end{aligned}$$

The general solution is thus

$$y(x) = \frac{1}{2} \cos^2(x) \sin(x) - \frac{1}{2} x \cos(x) + \frac{1}{4} \sin^3(x) + c_1 \cos(x) + c_2 \sin(x)$$

We now impose the initial conditions to fix  $c_1$  and  $c_2$

$$\begin{aligned} 1 &= y(0) = \frac{1}{2} \cos^2(0) \sin(0) - \frac{1}{2} 0 \cos(0) + \frac{1}{4} \sin^3(0) + c_1 \cos(0) + c_2 \sin(0) \\ &= 0 - 0 + 0 + c_1 + 0 \\ &= c_1 \end{aligned}$$

$$\begin{aligned} 2 &= y'(0) = \frac{1}{2} \cos^2(x) \sin(x) - \frac{1}{2} x \cos(x) + \frac{1}{4} \sin^3(x) + c_1 \cos(x) + c_2 \sin(x) \Big|_{x=0} \\ &= \frac{1}{2} (2 \cos(x) (-\sin(x)) \sin(x) + \cos^3(x)) - \frac{1}{2} (\cos(x) - x \sin(x)) + \frac{1}{4} (3 \sin^2(x) \cos(x) - c_1 \sin(x) + c_2 \cos(x)) \Big|_{x=0} \\ &= 0 + \frac{1}{2} - \frac{1}{2} + 0 + 0 - 0 + c_2 \\ &= c_2 \end{aligned}$$

Thus,  $c_1 = 1$ ,  $c_2 = 2$  and the solution to the initial value problem is

$$\frac{1}{2} \cos^2(x) \sin(x) - \frac{1}{2} x \cos(x) + \frac{1}{4} \sin^3(x) + \cos(x) + 2 \sin(x)$$

(c)  $y'' - 7y' + 10y = 100x$

- The corresponding homogeneous equation

(3)  $y'' - 7y' + 10y = 0$

is second order linear with constant coefficients and has

$$\lambda^2 - 7\lambda + 10 = 0$$

as its characteristic equation. Since  $\lambda^2 - 7\lambda + 10 = (\lambda - 2)(\lambda - 5)$  it's clear that we have two real roots  $\lambda = 2, 5$  and two linearly independent solutions of (3)

$$\begin{aligned} y_1(x) &= e^{2x} \\ y_2(x) &= e^{5x} \end{aligned}$$

The Wronskian for these two solutions is

$$W[y_1, y_2](x) = e^{2x} (5e^{5x}) - (2e^{2x}) e^{5x} = 3e^{7x}$$

Taking  $g(x) = 100x$  we can now plug into the Variation of Parameters formula

$$\begin{aligned} y_p(x) &= -y_1(x) \int^x \frac{y_2(s)g(s)}{W[y_1, y_2](s)} ds + y_2(x) \int^x \frac{y_1(s)g(s)}{W[y_1, y_2](s)} ds \\ &= -e^{5x} \int^x \frac{e^{2s} (100s)}{3e^{7s}} ds + e^{2x} \int^x \frac{e^{5s} (100s)}{3e^{7s}} ds \\ &= -\frac{100}{3} e^{5x} \int^x s e^{-5s} ds + \frac{100}{3} e^{2x} \int^x s e^{-2s} ds \\ &= -\frac{100}{3} e^{5x} \left( -\frac{1}{5} x e^{-5x} - \frac{1}{25} e^{-5x} \right) + \frac{100}{3} e^{2x} \left( -\frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2x} \right) \\ &= \frac{20}{3} x + \frac{4}{3} - \frac{50}{3} x - \frac{25}{3} \\ &= -10x - 7 \end{aligned}$$

The general solution is thus

$$y(x) = -10x - 7 + c_1 e^{5x} + c_2 e^{2x}$$

(d)  $y'' + 4y = \sec(2x)$

- The corresponding homogeneous equation

$$(4) \quad y'' + 4y = 0$$

is second order linear with constant coefficients and has

$$\lambda^2 + 4 = 0$$

as its characteristic equation. Since  $\lambda^2 + 4 = (\lambda - 2i)(\lambda + 2i)$  it's clear that we have two complex roots  $\lambda = \pm 2i$  and two linearly independent solutions of (4)

$$\begin{aligned} y_1(x) &= \cos(2x) \\ y_2(x) &= \sin(2x) \end{aligned}$$

Noting that

$$W[y_1, y_2](x) = \cos(2x)(2\cos(2x)) - (-2\sin(2x))\sin(2x) = 2(\cos^2(2x) + \sin^2(2x)) = 2$$

and

$$g(x) = \sec(2x) = \frac{1}{\cos(2x)}$$

we can now plug into the Variation of Parameters formula

$$\begin{aligned} y_p(x) &= -y_1(x) \int^x \frac{y_2(s)g(s)}{W[y_1, y_2](s)} ds + y_2(x) \int^x \frac{y_1(s)g(s)}{W[y_1, y_2](s)} ds \\ &= -\cos(2x) \int^x \frac{\sin(2s)}{2\cos(2s)} ds + \sin(2x) \int^x \frac{\cos(2s)}{2\cos(2s)} ds \\ &= -\cos(2x) \left( -\frac{1}{4} \ln(\cos 2x) \right) + \sin(2x) \left( \frac{1}{2} x \right) \\ &= \frac{1}{4} \cos(2x) \ln(\cos(2x)) + \frac{1}{2} x \sin(2x) \end{aligned}$$

The general solution of the original inhomogeneous differential equation is thus

$$y(x) = \frac{1}{4} \cos(2x) \ln(\cos(2x)) + \frac{1}{2} x \sin(2x) + c_1 \cos(2x) + c_2 \sin(2x)$$

4. Use the method of Variation of Parameters to solve the following non-homogeneous Euler-type equation.

$$x^2 y'' - 5xy' + 9y = x^3$$

- First we solve the corresponding homogeneous equation

$$x^2 y'' - 5xy' + 9y = 0$$

Substituting  $y(x) = x^r$  into this equation yields

$$0 = r(r-1)x^r - 5rx^r + 9x^r = (r-6r+9)x^r$$

So we must have

$$0 = r^2 - 6r + 9 = (r-3)^2$$

We thus have a single real root of the indicial equation;  $r = 3$ . The corresponding linearly independent real-valued solutions of the original Euler-type differential equation are

$$\begin{aligned} y_1(x) &= x^3 \\ y_2(x) &= x^3 \ln|x| \end{aligned}$$

The Wronskian of  $y_1(x)$  and  $y_2(x)$  is

$$W[y_1, y_2](x) = x^3 \left( 3x^2 \ln|x| + x^3 \left( \frac{1}{x} \right) - (3x^2) x^3 \ln|x| \right) = 3x^5 \ln|x| + x^5 - 3x^5 \ln|x| = x^5$$

To identify the function  $g(x)$  in the Variation of Parameters formula we must first cast the original non-homogeneous equation into standard form

$$y'' - \frac{5}{x}y' + \frac{9}{x^2}y = x.$$

Hence,  $g(x) = x$ . Thus we have

$$\begin{aligned} y_p(x) &= -y_1(x) \int^x \frac{y_2(s)g(s)}{W[y_1, y_2](s)} ds + y_2(x) \int^x \frac{y_1(s)g(s)}{W[y_1, y_2](s)} ds \\ &= -x^3 \int^x \frac{(s^3 \ln |s|) s}{s^5} ds + x^3 \ln |x| \int^x \frac{(s^3) s}{s^5} ds \\ &= -x^3 \int^x \ln |s| (s^{-1} ds) + x^3 \ln |x| \int^x \frac{1}{s} ds \\ &= -x^3 \int^{u=\ln |x|} u du + x^3 \ln |x| (\ln |x|) \\ &= -x^3 \left( \frac{1}{2} (\ln |x|)^2 \right) + x^3 (\ln |x|)^2 \\ &= \frac{1}{2} x^3 (\ln |x|)^2 \end{aligned}$$

Thus the general solution is

$$y(x) = y_p(x) + c_1 y_1(x) + c_2 y_2(x) = \frac{1}{2} x^3 (\ln |x|)^2 + c_1 x^3 + c_2 x^3 \ln |x|$$