

Math 2233
Homework Set 5

1. Determine whether the given equation is linear or nonlinear. If it is linear, write it in standard form and state whether it is homogeneous or non-homogeneous.

(a) $xy'' + 2x^3y' + y = 0$

- The equation is linear. To put it in standard form we divide through by x to get

$$y'' + 2x^2y' + \frac{1}{x} = 0.$$

This is a homogeneous linear differential equation. □

(b) $y'' + xy' + y^2 = 2x$

- This equation is non-linear due to the presence of the y^2 term. □

(c) $3y'' + 2y' + y = x^5$

- This equation is linear. To put it in standard form we divide through by 3 to get

$$y'' + \frac{2}{3}y' + \frac{1}{3}y = \frac{1}{3}x^5.$$

This equation is non-homogeneous, due to the presence of the $\frac{1}{3}x^5$ on the right hand side when it's written in standard form. □

2. Verify that the two given functions are linearly independent solutions of the given homogeneous equation and then find the general solution.

(a) $y'' + 9y = 0$, $y_1(x) = \sin(3x)$, $y_2(x) = \cos(3x)$

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$$\begin{aligned}y_1'' + 9y_1 &= (3 \cos(3x))' + 9 \sin(3x) \\ &= 3(-3 \sin(3x)) + 9 \sin(3x) \\ &= (-9 + 9) \sin(3x) \\ &= 0\end{aligned}$$

$$\begin{aligned}y_2'' + 9y_2 &= (-3 \sin(3x))' + 9 \cos(3x) \\ &= -3(3 \cos(3x)) + 9 \cos(3x) \\ &= (-9 + 9) \cos(3x) \\ &= 0\end{aligned}$$

So both y_1 and y_2 are solutions. They are linearly independent since

$$\begin{aligned}W[y_1, y_2](x) &\equiv y_1(x)y_2'(x) - y_1'(x)y_2(x) \\ &= \sin(3x)(-3 \sin(3x)) - (3 \cos(3x)) \cos(3x) \\ &= -3(\sin^2(3x) + \cos^2(3x)) \\ &= -3 \cdot 1 \\ &= -3 \\ &\neq 0\end{aligned}$$

Since $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the homogeneous linear equation $y'' + 9y = 0$, the general solution of this equation is

$$\begin{aligned} y(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \sin(3x) + c_2 \cos(3x) \end{aligned}$$

□

(b) $y'' + 2y' - 15y = 0$, $y_1(x) = e^{3x}$, $y_2(x) = e^{-5x}$

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$$\begin{aligned} y_1'' + 2y_1' - 15y_1 &= (3)(3)e^{3x} + 2(3e^{3x}) - 15e^{3x} \\ &= (9 + 6 - 15)e^{3x} \\ &= 0 \end{aligned}$$

$$\begin{aligned} y_2'' + 2y_2' - 15y_2 &= (-5)(-5)e^{-5x} + 2(-5e^{-5x}) - 15e^{-5x} \\ &= (25 - 10 - 15)e^{-5x} \\ &= 0 \end{aligned}$$

So both $y_1(x)$ and $y_2(x)$ are solutions. They are also linearly independent since

$$\begin{aligned} W[y_1, y_2](x) &\equiv y_1(x)y_2'(x) - y_1'(x)y_2(x) \\ &= e^{3x}(-5e^{-5x}) - (3e^{3x})e^{-5x} \\ &= (-5 - 3)e^{-2x} \\ &= -8e^{-3x} \\ &\neq 0 \end{aligned}$$

Since $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the homogeneous linear equation $y'' + 2y' - 15y = 0$, the general solution of this equation is

$$\begin{aligned} y(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 e^{3x} + c_2 e^{-5x} \end{aligned}$$

□

(c) $y'' + 4y' + 4y = 0$, $y_1(x) = e^{-2x}$, $y_2(x) = xe^{-2x}$

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$$\begin{aligned} y_1'' + 4y_1' + 4y_1 &= (-2)(-2)e^{-2x} + 4(-2e^{-2x}) + 4e^{-2x} \\ &= (4 - 8 + 4)e^{-2x} \\ &= 0 \end{aligned}$$

$$\begin{aligned} y_2'' + 4y_2' + 4y_2 &= (e^{-2x} - 2xe^{-2x})' + 4(e^{-2x} - 2xe^{-2x}) + 4xe^{-2x} \\ &= (-2e^{-2x} - 2e^{-2x} + 4xe^{-2x}) + 4(e^{-2x} - 2xe^{-2x}) + 4xe^{-2x} \\ &= (-2 - 2 + 4)e^{-2x} + (4 - 8 + 4)xe^{-2x} \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

So both $y_1(x)$ and $y_2(x)$ are solutions. They are also linearly independent since

$$\begin{aligned} W[y_1, y_2](x) &\equiv y_1(x)y_2'(x) - y_1'(x)y_2(x) \\ &= e^{-2x} (e^{-2x} - 2xe^{-2x}) - (-2e^{-2x}) xe^{-2x} \\ &= e^{-4x} - 2xe^{-4x} + 2xe^{-2x} \\ &= e^{-4x} \\ &\neq 0 \end{aligned}$$

Since $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the homogeneous linear equation $y'' + 4y' + 4y = 0$, the general solution of this equation is

$$\begin{aligned} y(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1e^{-2x} + c_2xe^{-2x} \end{aligned}$$

□

3. Given that $y_1(x) = e^{3x}$ is one solution of $y'' - 5y' + 6y = 0$, find a second linearly independent solution and then write down the general solution.

- To find a second linearly independent solution we apply the Reduction of Order formula

$$y_2(x) = y_1(x) \int \frac{1}{[y_1(s)]^2} \exp \left[- \int^s p(t) dt \right] ds$$

For the case at hand, we have $y_1(x) = e^{3x}$ and $p(x) = -5$, so

$$\begin{aligned} y_2(x) &= e^{3x} \int \frac{1}{(e^{3s})^2} \exp \left[- \int^s (-5) dt \right] ds \\ &= e^{3x} \int e^{-6s} \exp [5s] ds \\ &= e^{3x} \int e^{-6s} e^{5s} ds \\ &= e^{3x} \int e^{-s} ds \\ &= e^{3x} (-e^{-x}) \\ &= -e^{2x} \\ &\sim e^{2x} \end{aligned}$$

In the last step we dropped the minus sign simply because if $-e^{2x}$ is a solution so is e^{2x} (because of the Superposition Principle), and the latter expression for $y_2(x)$ is a tad bit simpler.

The general solution is a linear combination of $y_1(x)$ and $y_2(x)$:

$$y(x) = c_1e^{3x} + c_2e^{2x}.$$

4. Given that $y_1(x) = e^{2x}$ is one solution of $y'' - 4y = 0$, find a second linearly independent solution and then write down the general solution.

- We'll again apply the Reduction of Order formula to get a second linearly independent solution. For this problem we have $y_1(x) = e^{2x}$ and $p(x) = 0$. So

$$\begin{aligned}
 y_2(x) &= e^{2x} \int^x \frac{1}{(e^{2s})^2} \exp \left[- \int^s 0 \cdot dt \right] ds \\
 &= e^{2x} \int^x e^{-4s} e^0 ds \\
 &= e^{2x} \int^x e^{-4s} ds \\
 &= e^{2x} \left(-\frac{1}{4} e^{-4x} \right) \\
 &= -\frac{1}{4} e^{-2x} \\
 &\sim e^{-2x}
 \end{aligned}$$

Again we have thrown away a constant factor to simplify the expression of $y_2(x)$. The general solution is thus

$$y(x) = c_1 e^{2x} + c_2 e^{-2x}.$$

5. Given that $y_1(x) = x$ is one solution of $y'' - 2xy' + 2y = 0$, find a second linearly independent solution and then write down the general solution.

- We'll again apply the Reduction of Order formula to get a second linearly independent solution. For this problem we have $y_1(x) = x$ and $p(x) = -2x$. So

$$\begin{aligned}
 y_2(x) &= x \int^x \frac{1}{s^2} \exp \left[\int^s 2tdt \right] ds \\
 &= x \int^x \frac{1}{s^2} \exp [s^2] ds \\
 &= x \int^x \frac{e^{s^2}}{s^2} ds
 \end{aligned}$$

Unfortunately, we can not actually carry out the final integration to get a simple formula of $y_2(x)$. Nevertheless, the integral can at least always be evaluated numerically, and we can write the following formula for the general solution of the original differential equation

$$y(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 x + c_2 x \int^x \frac{e^{s^2}}{s^2} ds.$$

6. Given that $y_1(x) = x \sin(x)$ is one solution of $x^2 y'' - 2xy' + (x^2 + 2)y = 0$, find a second linearly independent solution and then write down the general solution.

- We'll again apply the Reduction of Order formula to get a second linearly independent solution. This time we have $y_1(x) = x \sin(x)$ and $p(x) = -\frac{2}{x}$ (to identify $p(x)$ we first put the differential equation

in standard form). So

$$\begin{aligned}
 y_2(x) &= x \sin(x) \int^x \frac{1}{(s \sin(s))^2} \exp \left[- \int^s -\frac{2}{t} dt \right] ds \\
 &= x \sin(x) \int^x \frac{1}{(s \sin(s))^2} \exp [2 \ln |s|] ds \\
 &= x \sin(x) \int^x \frac{1}{(s \sin(s))^2} s^2 ds \\
 &= x \sin(x) \int^x \frac{1}{\sin^2(s)} ds \\
 &= x \sin(x) \int^x \csc^2(s) ds \\
 &= x \sin(x) (-\cot(x)) \\
 &= x \sin(x) \left(-\frac{\cos(x)}{\sin(x)} \right) \\
 &= -x \cos(x) \\
 &\sim x \cos(x)
 \end{aligned}$$

where in the last step we dropped the factor of -1 to simplify the expression for $y_2(x)$.

The general solution is thus

$$y(x) = c_1 x \sin(x) + c_2 x \cos(x).$$

7. Find the general solution of the following differential equations

(a) $y'' - 5y = 0$.

- This is a second order linear equation with constant coefficients and so we look for solutions of the form $y(x) = e^{\lambda x}$. Plugging $y(x) = e^{\lambda x}$ into the differential equation yields

$$\lambda^2 e^{\lambda x} - 5e^{\lambda x} = 0$$

or

$$(\lambda^2 - 5) e^{\lambda x} = 0$$

Thus the characteristic equation for this differential equation is

$$\lambda^2 - 5 = 0$$

or

$$\lambda^2 = 5$$

which obviously has as solutions

$$\lambda = \pm\sqrt{5}$$

So both $y_1(x) = e^{\sqrt{5}x}$ and $y_2(x) = e^{-\sqrt{5}x}$ are solutions of the differential equation. Moreover, they are linearly independent since

$$W[y_1, y_2](x) = e^{\sqrt{5}x} \left(-\sqrt{5}e^{-\sqrt{5}x} \right) - \left(\sqrt{5}e^{\sqrt{5}x} \right) e^{-\sqrt{5}x} = -2\sqrt{5} \neq 0.$$

Therefore the general solution of the differential equation is

$$y(x) = c_1 e^{\sqrt{5}x} + c_2 e^{-\sqrt{5}x}.$$

(b) $y'' - 3y + 2y = 0$

- The characteristic equation for this second order linear equation with constant coefficients is

$$\lambda^2 - 3\lambda + 2 = 0$$

or

$$(\lambda - 1)(\lambda - 2) = 0$$

Thus, we have $\lambda = 1, 2$ as solutions. To each of these roots of the characteristic equation we have a corresponding solution of the original differential equation; namely, $y_1(x) = e^x$ and $y_2(x) = e^{2x}$. These two solutions are linearly independent and so the general solution is

$$y(x) = c_1e^x + c_2e^{2x}.$$

(c) $y'' - y' - 20y = 0$

- The characteristic equation for this homogeneous second order linear equation with constant coefficients is

$$\lambda^2 - \lambda - 20 = 0$$

or

$$(\lambda - 5)(\lambda + 4) = 0.$$

Thus, $\lambda = 5, -4$ and we have two linearly independent solutions $y_1(x) = e^{5x}$ and $y_2(x) = e^{-4x}$. The general solution is thus

$$y(x) = c_1e^{5x} + c_2e^{-4x}.$$

(d) $y'' - 13y' + 42y = 0$

- The characteristic equation for this homogeneous second order linear equation with constant coefficients is

$$\lambda^2 - 13\lambda + 42 = 0$$

To solve the characteristic equation we apply the Quadratic Formula:

$$\begin{aligned} \lambda &= \frac{-(-13) \pm \sqrt{(-13)^2 - 4(42)}}{2} \\ &= \frac{13 \pm \sqrt{169 - 168}}{2} \\ &= \frac{13 \pm 1}{2} \\ &= 7, 6 \end{aligned}$$

Thus, $\lambda = 6, 7$ and we have two linearly independent solutions $y_1(x) = e^{6x}$ and $y_2(x) = e^{7x}$. The general solution is thus

$$y(x) = c_1e^{6x} + c_2e^{7x}$$

(e) $y'' + y' + 7y = 0$

- The characteristic equation for this homogeneous second order linear equation with constant coefficients is

$$\lambda^2 + \lambda + 7 = 0$$

To solve the characteristic equation we apply the Quadratic Formula:

$$\begin{aligned}\lambda &= \frac{-1 \pm \sqrt{(1)^2 - (4)(7)}}{2} \\ &= \frac{-1 \pm \sqrt{1 - 28}}{2} \\ &= \frac{-1 \pm \sqrt{-27}}{2} \\ &= \frac{-1 \pm \sqrt{27}i}{2}\end{aligned}$$

Thus, we have a pair of complex roots

$$\lambda = -\frac{1}{2} + \frac{\sqrt{27}}{2}i, -\frac{1}{2} - \frac{\sqrt{27}}{2}i$$

and so we take

$$\begin{aligned}\alpha &= \operatorname{Re}(\lambda) = -\frac{1}{2} \\ \beta &= \pm \operatorname{Im}(\lambda) = \frac{\sqrt{27}}{2}\end{aligned}$$

Associated to the complex roots $\lambda = \alpha \pm i\beta$ are the following real-valued solutions of the original differential equation:

$$\begin{aligned}y_1(x) &= e^{\alpha x} \cos(\beta x) = e^{-\frac{1}{2}x} \cos\left(\frac{\sqrt{27}}{2}x\right) \\ y_2(x) &= e^{\alpha x} \sin(\beta x) = e^{-\frac{1}{2}x} \sin\left(\frac{\sqrt{27}}{2}x\right)\end{aligned}$$

and so the general solution is

$$y(x) = c_1 e^{-\frac{1}{2}x} \cos\left(\frac{\sqrt{27}}{2}x\right) + c_2 e^{-\frac{1}{2}x} \sin\left(\frac{\sqrt{27}}{2}x\right).$$

(f) $y'' + 2y' + 5y = 0$

- The characteristic equation for this homogeneous second order linear equation with constant coefficients is

$$\lambda^2 + 2\lambda + 5 = 0$$

To solve the characteristic equation we apply the Quadratic Formula:

$$\begin{aligned}\lambda &= \frac{-2 \pm \sqrt{(2)^2 - (4)(5)}}{2} \\ &= \frac{-2 \pm \sqrt{4 - 20}}{2} \\ &= \frac{-2 \pm \sqrt{-16}}{2} \\ &= \frac{-2 \pm 4i}{2} \\ &= -1 \pm 2i\end{aligned}$$

Thus, we have a pair of complex roots

$$\lambda = -1 + 2i, -1 - 2i$$

so we take $\alpha = -1 = \operatorname{Re}(\lambda)$ and $\beta = 2 = \pm \operatorname{Im}(\lambda)$. Associated to this pair of complex roots are the following real-valued solutions of the original differential equation:

$$\begin{aligned}y_1(x) &= e^{\alpha x} \cos(\beta x) = e^{-x} \cos(2x) \\y_2(x) &= e^{\alpha x} \sin(\beta x) = e^{-x} \sin(2x)\end{aligned}$$

and so the general solution is

$$y(x) = c_1 e^{-x} \cos(2x) + c_2 e^{-x} \sin(2x).$$

8. Solve the following initial value problems.

(a) $y'' - 9y = 0$, $y(0) = 1$, $y'(0) = 2$.

- The characteristic equation for this homogeneous second order linear equation with constant coefficients is

$$\lambda^2 - 9 = 0$$

or

$$(\lambda - 3)(\lambda + 3) = 0$$

so $\lambda = 3, -3$. The general solution is thus

$$y(x) = c_1 e^{3x} + c_2 e^{-3x}.$$

We now impose the initial conditions to fix the constants c_1 and c_2 .

$$\begin{aligned}1 &= y(0) = c_1 e^0 + c_2 e^0 = c_1 + c_2 \\2 &= y'(0) = 3c_1 e^{3x} - 3c_2 e^{-3x} \Big|_{x=0} = 3c_1 - 3c_2\end{aligned}$$

Thus we have two equations and two unknowns

$$\begin{aligned}c_1 + c_2 &= 1 \\3c_1 - 3c_2 &= 2\end{aligned}$$

Adding 3 times the first equation to the second equation yields

$$6c_1 + 0 = 5$$

so $c_1 = \frac{5}{6}$. But then the equation $c_1 + c_2 = 1$ implies that $c_2 = \frac{1}{6}$. Thus, the solution to the initial value problem is

$$y(x) = \frac{5}{6} e^{3x} + \frac{1}{6} e^{-3x}.$$

(b) $y'' - 2y' + y = 0$, $y(0) = 2$, $y'(0) = 1$.

- he characteristic equation for this homogeneous second order linear equation with constant coefficients is

$$\lambda^2 - 2\lambda + 1 = 0$$

or

$$(\lambda - 1)^2 = 0$$

so $\lambda = 1$. So $y_1(x) = e^x$ is one solution of the differential equation, and (because we are in the case where there is only one distinct root for the characteristic equation) $y_2(x) = xy_1(x) = xe^x$ is a second linearly independent solution. The general solution is thus

$$y(x) = c_1 e^x + c_2 x e^x.$$

We now impose the initial conditions to fix c_1 and c_2 :

$$\begin{aligned}2 &= y(0) = c_1 e^0 + c_2(0)e^0 = c_1 \\1 &= y'(0) = c_1 e^x + c_2(e^x + x e^x) \Big|_{x=0} = c_1 + c_2\end{aligned}$$

Thus, we must have

$$\begin{aligned}c_1 &= 2 \\c_1 + c_2 &= 1\end{aligned}$$

Obviously, we must have $c_1 = 2$ and $c_2 = -1$. Thus, the solution to the initial value problem is

$$y(x) = 2e^x - xe^x.$$

(c) $y'' + 2y' + 2y = 0$, $y(0) = 1$, $y'(0) = -1$

- The characteristic equation for this homogeneous second order linear equation with constant coefficients is

$$\lambda^2 + 2\lambda + 2 = 0$$

The Quadratic Formula thus implies

$$\begin{aligned}\lambda &= \frac{-2 \pm \sqrt{(2)^2 - (4)(2)}}{2} \\&= \frac{-2 \pm \sqrt{-4}}{2} \\&= \frac{-2 \pm 2i}{2} \\&= -1 \pm i\end{aligned}$$

We thus have a pair of complex roots $\lambda = \alpha \pm i\beta$ with $\alpha = -1$ and $\beta = 1$. The general solution is thus

$$y(x) = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x) = c_1 e^x \cos(x) + c_2 e^x \sin(x).$$

We now impose the initial conditions to fix c_1 and c_2 :

$$\begin{aligned}1 &= y(0) = c_1 e^0 \cos(0) + c_2 e^0 \sin(0) = c_1(1)(1) + c_2(1)(0) = c_1 \\-1 &= y'(0) = c_1 (e^x \cos(x) - e^x \sin(x)) + c_2 (e^x \sin(x) + e^x \cos(x))|_{x=0} = c_1(1 - 0) + c_2(0 + 1) = c_1 + c_2\end{aligned}$$

We thus have

$$\begin{aligned}c_2 &= 1 \\c_1 + c_2 &= -1\end{aligned}$$

which implies $c_1 = -2$ and $c_2 = 1$. Thus, the solution to the initial value problem is

$$y(x) = -2e^x \cos(x) + e^x \sin(x).$$