Math 2233
Homework Set 4

1. Verify that each of the following differential equations is exact and then find the general solution.

(a) $2xy\,dx + \left(x^2+1\right)dy = 0$

- $$\begin{aligned} M &= 2xy \\ N &= x^2+1 \\ \frac{\partial M}{\partial y} &= 2x = \frac{\partial N}{\partial x} \quad \Rightarrow \quad Exact \end{aligned}$$

  Since the differential equation is exact it is equivalent to an algebraic relation of the form

  $$\phi(x,y) = C$$

  with

  $$\frac{\partial \phi}{\partial x} = M = 2xy \tag{1}$$
  $$\frac{\partial \phi}{\partial y} = N = x^2+1 \tag{2}$$

  The most general function ϕ satisfying (1) is obtained taking the anti-partial dervivative with respect to x; i.e., by integrating with respect to x, treating y as a constant, and allowing the possibility of an arbitrary function of y in the result:

  $$\phi(x,y) = \int \frac{\partial \phi}{\partial x}\partial x = \int (2xy)\partial x = yx^2 + H_1(y)$$

  Similarly, the most general function ϕ satisfying (2) is

  $$\phi(x,y) = \int \frac{\partial \phi}{\partial y}\partial y = \int (x^2+1)\partial y = x^2y + y + H_2(x)$$

  Comparing these two expressions for $\phi(x,y)$ and demanding that they agree with one another, we see that we must take

  $$\begin{aligned} H_1(y) &= y \\ H_2(x) &= 0 \end{aligned}$$

  Hence, $\phi(x,y) = x^2y + y$ and our differential equation is equivalent to the following family of algebraic relations

  $$x^2y + y = C \quad , \quad \text{with } C \text{ an arbitrary constant .}$$

  Solving this relation for y yields

  $$y(x) = \frac{C}{x^2+1}.$$

(b) $3x^2y\,dx + \left(x^3+1\right)dy = 0$

- $$\begin{aligned} M &= 3x^2y \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 3x^2 \\ N &= x^3+1 \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 3x^2 \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ the equation is exact.

$$\begin{aligned}
\phi(x,y) &= \int \frac{\partial \phi}{\partial x}\partial x = \int M(x,y)\partial x = \int 3x^2 y \partial x = x^3 y + H_1(y) \\
\phi(x,y) &= \int \frac{\partial \phi}{\partial y}\partial y = \int N(x,y)\partial y = \int \left(x^3+1\right)\partial y = x^3 y + y + H_2(x)
\end{aligned}$$

Comparing these two expressions for $\phi(x,y)$ we see that we must take $H_1(y) = y$ and $H_2(x) = 0$. So $\phi(x,y) = x^3 y + y$ and the differential equation is equivalent to

$$x^3 y + y = C$$

or

$$y(x) = \frac{C}{1+x^3}$$

(c) $y(y+2x)dx + x(2y+x)dy = 0$

•

$$\begin{aligned}
M &= y^2 + 2yx \;\Rightarrow\; \frac{\partial M}{\partial y} = 2y + 2x \\
N &= 2yx + x^2 \;\Rightarrow\; \frac{\partial N}{\partial x} = 2y + 2x
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ the equation is exact.

$$\begin{aligned}
\phi(x,y) &= \int \frac{\partial \phi}{\partial x}\partial x = \int M(x,y)\partial x = \int (y^2 + 2yx)\partial x = y^2 x + yx^2 + H_1(y) \\
\phi(x,y) &= \int \frac{\partial \phi}{\partial y}\partial y = \int N(x,y)\partial y = \int \left(2yx + x^2\right)\partial y = y^2 x + x^2 y + H_2(x)
\end{aligned}$$

Comparing these two expressions for $\phi(x,y)$ we see that we must take $H_1(y) = 0$ and $H_2(x) = 0$. So $\phi(x,y) = x^2 y + y^2 x$ and the differential equation is equivalent to

$$x^2 y + y^2 x = C$$

Solving this equation for y yields

$$y = \frac{1}{2x}\left(-x^2 \pm \sqrt{(x^4 + 4xC)}\right).$$

(d) $y\cos(xy)\,dx + x\cos(xy)\,dy = 0$

•

$$\begin{aligned}
M &= y\cos(xy) \;\Rightarrow\; \frac{\partial M}{\partial y} = \cos(xy) - xy\sin(xy) \\
N &= x\cos(xy) \;\Rightarrow\; \frac{\partial N}{\partial x} = \cos(xy) - xy\sin(xy)
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ the equation is exact.

$$\begin{aligned}
\phi(x,y) &= \int \frac{\partial \phi}{\partial x}\partial x = \int M(x,y)\partial x = \int y\cos(xy)\partial x = \sin(xy) + H_1(y) \\
\phi(x,y) &= \int \frac{\partial \phi}{\partial y}\partial y = \int N(x,y)\partial y = \int x\cos(xy)\partial y = \sin(xy) + H_2(x)
\end{aligned}$$

Comparing these two expressions for $\phi(x,y)$ we see that we must take $H_1(y) = 0$, $H_2(x) = 0$, and $\phi(x,y) = \sin(xy)$. Thus the original differential equation is equivalent to

$$\sin(xy) = C$$

or

$$y = \frac{C'}{x}$$

(Here $C' = \sin^{-1}(C)$.)

2. Solve the following initial value problems.

(a) $(x - y\cos(x)) - \sin(x)y' = 0$, $y\left(\frac{\pi}{2}\right) = 1$

- This equation is exact since

$$\frac{\partial}{\partial y}(x - y\cos(x)) = -\cos(x) = \frac{\partial}{\partial x}(\sin(x))$$

Therefore, it must be equivalent to an algebraic equation of the form $\phi(x,y) = C$ with

$$\begin{aligned}\phi(x,y) &= \int \frac{\partial \phi}{\partial x}\partial x = \int M(x,y)\partial x = \int (x - y\cos(x))\,\partial x = \frac{1}{2}x^2 - y\sin(x) + H_1(y) \\ \phi(x,y) &= \int \frac{\partial \phi}{\partial y}\partial y = \int N(x,y)\partial y = \int (-\sin(x))\,\partial y = -y\sin(x) + H_2(x)\end{aligned}$$

Comparing these two expressions for $\phi(x,y)$ we see we must take $H_1(y) = 0$, $H_2(x) = \frac{1}{2}x^2$, and $\phi(x,y) = \frac{1}{2}x^2 - y\sin(x)$. Hence we must have

$$\frac{1}{2}x^2 - y\sin(x) = C.$$

Before solving for y we'll impose the initial condition: $x = \frac{\pi}{2} \Rightarrow y = 1$ to first determine C.

$$C = \frac{1}{2}x^2 - y\sin(x) = \frac{1}{2}\left(\frac{\pi}{2}\right)^2 - (1)\sin\left(\frac{\pi}{2}\right) = \frac{1}{8}\pi^2 - 1.$$

Now we solve for y :

$$y = \frac{\frac{1}{2}x^2 - C}{\sin(x)} = \csc(x)\left(\frac{1}{2}x^2 + 1 - \frac{1}{8}\pi^2\right)$$

(b) $x^2 + y^2 + 2xyy' = 0$, $y(1) = 1$

- This equation is exact since

$$\frac{\partial}{\partial y}\left(x^2 + y^2\right) = 2y = \frac{\partial}{\partial x}(2xy).$$

Therefore, the differential equation is equivalent to an algebraic relation of the form $\phi(x,y) = C$ with

$$\begin{aligned}\phi(x,y) &= \int \frac{\partial \phi}{\partial x}\partial x = \int M(x,y)\partial x = \int \left(x^2 + y^2\right)\partial x = \frac{1}{3}x^3 - xy^2 + H_1(y) \\ \phi(x,y) &= \int \frac{\partial \phi}{\partial y}\partial y = \int N(x,y)\partial y = \int (2xy)\,\partial y = xy^2 + H_2(x)\end{aligned}$$

Comparing these two expressions for $\phi(x,y)$ we see we must take $H_1(y) = 0$, $H_2(x) = \frac{1}{3}x^3$, and so $\phi(x,y) = \frac{1}{3}x^3 - xy^2$. We thus have

$$\frac{1}{3}x^3 + xy^2 = C.$$

We now impose the initial condition $x = 1 \Rightarrow y = 1$ to fix C:

$$C = \frac{1}{3}x^3 + xy^2 = \frac{1}{3}(1)^3 + (1)(1)^2 = \frac{4}{3}.$$

Hence, the differential equation together with the initial condition implies that y must satisfy

$$\frac{1}{3}x^3 + xy^2 = \frac{4}{3}.$$

Solving this equation for y yields

$$y = \pm\sqrt{\frac{1}{3x}\left(4 - x^3\right)}$$

3. Find an integrating factor for each of the following differential equations and obtain the general solution.

(a) $y + (y - x)y' = 0$

- Suppose $\mu(x, y)$ is an integrating factor for this equation. Then

$$\mu y + \mu(y - x)y' = 0$$

must be exact so

$$\frac{\partial}{\partial y}(\mu y) = \frac{\partial}{\partial x}(\mu y - \mu x)$$

or

$$\frac{\partial \mu}{\partial y} y + \mu = \frac{\partial \mu}{\partial x} y - \frac{\partial \mu}{\partial x} x - \mu$$

In order to simplify this equation we make the hypothesis that $\mu(x, y) = \mu(y)$. Then $\frac{\partial \mu}{\partial y} = \frac{d\mu}{dy}$ and $\frac{\partial \mu}{\partial x} = 0$; so we have

$$y\frac{d\mu}{dy} + \mu = -\mu$$

or

$$\frac{d\mu}{dy} + \frac{2}{y}\mu = 0.$$

This is a first order linear equation for μ. It is also separable, since we can rewrite it as

$$\frac{d\mu}{\mu} = -\frac{2dy}{y}$$

Integrating both sides yields

$$\ln|\mu| = -2\ln|y|$$

or

$$\mu = e^{-2\ln|y|} = e^{\ln\left|y^{-2}\right|} = y^{-2}.$$

Therefore y^{-2} should be our integrating factor. Multiplying the original differential equation by y^{-2} yields

$$\frac{1}{y} + \left(\frac{1}{y} - \frac{x}{y^2}\right)y' = 0$$

Now we have

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{1}{y}\right) = -\frac{1}{y^2} \\
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y} - \frac{x}{y^2}\right) = -\frac{1}{y^2}
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ the equation is exact. Therefore the differential equation is equivalent to a (family of) algebraic equation(s) of the form

$$\phi(x, y) = C$$

with

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M = \frac{1}{y} \\
\frac{\partial \phi}{\partial y} &= N = \frac{1}{y} - \frac{x}{y^2}
\end{aligned}$$

Taking anti-partial derivatives of both these equations yields the following two expressions for ϕ;

$$\begin{aligned}
\phi(x,y) &= \int \frac{\partial\phi}{\partial x}\partial x = \int \left(\frac{1}{y}\right)\partial x = \frac{x}{y} + H_1(y) \\
\phi(x,y) &= \int \frac{\partial\phi}{\partial y}\partial y = \int \left(\frac{1}{y} - \frac{x}{y^2}\right)\partial y = \ln|y| + \frac{x}{y} + H_2(x)
\end{aligned}$$

Comparing these two expressions we see we must take $H_1(y) = \ln|y|$, $H_2(x) = 0$, and so $\phi(x,y) = \ln|y| + \frac{x}{y}$. Hence, an implicit solution to the original differential equation is

$$\ln|y| + \frac{x}{y} = C.$$

(b) $x^2 + y^2 + x + yy' = 0$

- If $\mu(x,y)$ is a integrating factor for this equation we must have

$$\frac{\partial}{\partial y}\left(\mu x^2 + \mu y^2 + \mu x\right) = \frac{\partial}{\partial x}(\mu y)$$

or

$$\frac{\partial\mu}{\partial y}x^2 + \frac{\partial\mu}{\partial y}y^2 + \mu(2y) + \frac{\partial\mu}{\partial y}x = \frac{\partial\mu}{\partial x}y$$

If we suppose $\mu(x,y)$ actually only depends on x then the above partial differential equation for μ simplifies to

$$2y\mu = y\frac{d\mu}{dx}$$

or

$$\frac{d\mu}{\mu} = 2dx$$

Integrating both sides yields

$$\ln|\mu| = 2x$$

or

$$\mu(x) = e^{2x}.$$

Hence, e^{2x} should be an integrating factor for the original differential equation. Hence,

$$x^2e^{2x} + y^2e^{2x} + xe^{2x} + ye^{2x}y' = 0$$

should be exact. Indeed,

$$\frac{\partial}{\partial y}\left(x^2e^{2x} + y^2e^{2x} + xe^{2x}\right) = 2ye^{2x} = \frac{\partial}{\partial x}\left(ye^{2x}\right)$$

so the equation is exact. Therefore we look for a function $\phi(x,y)$ such that

$$\begin{aligned}
\frac{\partial\phi}{\partial x} &= x^2e^{2x} + y^2e^{2x} + xe^{2x} \\
\frac{\partial\phi}{\partial y} &= ye^{2x}
\end{aligned}$$

Taking anti-partial derivatives of both these equations yields

$$\begin{aligned}
\phi(x,y) &= \int \left(x^2e^{2x} + y^2e^{2x} + xe^{2x}\right)\partial x = \frac{1}{2}x^2e^{2x} + \frac{1}{2}y^2e^{2x} + H_1(y) \\
\phi(x,y) &= \int ye^{2x}\partial y = \frac{1}{2}y^2e^{2x} + H_2(x)
\end{aligned}$$

Comparing these two expressions for $\phi(x,y)$ we see that we must take $H_1(y)=0$, $H_2(x)=\frac{1}{2}x^2e^{2x}$, and so $\phi(x,y)=\frac{1}{2}x^2e^{2x}+\frac{1}{2}y^2e^{2x}$.. Thus, the general (implicit) solution of the original differential equation is

$$\frac{1}{2}x^2e^{2x}+\frac{1}{2}y^2e^{2x}=C$$

Solving for y yields

$$y(x)=\pm\sqrt{2Ce^{-2x}-x^2}$$

(c) $2y^2+(2x+3xy)y'=0$

- If $\mu(x,y)$ is to be an integrating factor we must have

$$\frac{\partial}{\partial y}\left(2y^2\mu\right)=\frac{\partial}{\partial x}\left(2x\mu+3xy\mu\right)$$

or

$$4y\mu+2y^2\frac{\partial\mu}{\partial y}=2\mu+2x\frac{\partial\mu}{\partial x}+3y\mu+3xy\frac{\partial\mu}{\partial x}$$

This equation would simplify tremedously if we supposed that $\mu(x,y)$ actually depended only on y. Then we would have

$$4y\mu+2y^2\frac{d\mu}{dy}=2\mu+3y\mu$$

or

$$2y^2\frac{d\mu}{dy}=(2-y)\,\mu$$

or

$$\frac{d\mu}{\mu}=\left(\frac{1}{y^2}-\frac{1}{2y}\right)dy$$

Integrating both sides yields

$$\ln|\mu|=-\frac{1}{y}-\frac{1}{2}\ln|y|$$

or

$$\mu=\exp\left(-\frac{1}{y}+\ln|y^{-\frac{1}{2}}|\right)=\frac{1}{y^{1/2}}e^{-\frac{1}{y}}.$$

So $\frac{1}{\sqrt{y}}e^{-\frac{1}{y}}$ should be an integrating factor for the original differential equation. Indeed, multiplying the original equation by $\frac{1}{\sqrt{y}}e^{-\frac{1}{y}}$ yields

$$2y^{3/2}e^{-\frac{1}{y}}+\left(2\frac{x}{y^{1/2}}e^{-\frac{1}{y}}+3y^{1/2}xe^{-\frac{1}{y}}\right)y'$$

which is exact since

$$\frac{\partial}{\partial y}\left(2y^{3/2}e^{-\frac{1}{y}}\right)=3y^{1/2}e^{-\frac{1}{y}}+2y^{3/2}\left(+\frac{1}{y^2}e^{-\frac{1}{y}}\right)=3y^{1/2}e^{-\frac{1}{y}}+2y^{-1/2}e^{-\frac{1}{y}}$$

$$\frac{\partial}{\partial x}\left(2\frac{x}{y^{1/2}}e^{-\frac{1}{y}}+3y^{1/2}xe^{-\frac{1}{y}}\right)=2y^{-1/2}e^{-\frac{1}{y}}+3y^{1/2}e^{-\frac{1}{y}}$$

and so we have $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.Now we need to find a $\phi(x,y)$ such that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=2y^{3/2}e^{-\frac{1}{y}}\\ \frac{\partial\phi}{\partial y}&=2\frac{x}{y^{1/2}}e^{-\frac{1}{y}}+3y^{1/2}xe^{-\frac{1}{y}}\end{aligned}$$

Taking ant-partial derivatives of both of these equations yields

$$\begin{aligned}\phi(x,y) &= 2y^{3/2}e^{-\frac{1}{y}}x + H_1(y)\\ \phi(x,y) &= 2xy^{3/2}e^{-\frac{1}{y}} + H_2(x)\end{aligned}$$

Hence, we must have $H_1(y) = 0$, $H_2(x) = 0$, and $\phi(x,y) = 2y^{3/2}e^{-\frac{1}{y}}$. Thus, the general solution to the original differential equation is given implicitly by

$$2xy^{3/2}e^{-\frac{1}{y}} = C.$$

(d) $xy - x^2y' = 0$

- If $\mu(x,y)$ is to be an integrating factor we must have

$$\frac{\partial}{\partial y}(xy\mu) = \frac{\partial}{\partial x}\left(-x^2\mu\right)$$

or

$$x\mu + xy\frac{\partial\mu}{\partial y} = -2x\mu - x^2\frac{\partial\mu}{\partial x}.$$

This equation simplifies tremendously if we can assume μ depends only on x. In this case, the partial differential equation for μ becomes

$$x\mu + 0 = -2x\mu - x^2\frac{d\mu}{dx}$$

or

$$x^2\frac{d\mu}{dx} = -3x\mu$$

or

$$\frac{d\mu}{\mu} = -\frac{3}{x}dx$$

Integrating both sides yieldsx

$$\ln|\mu| = -3\ln|x|. = \ln|x^{-3}|$$

or $\mu = x^{-3}$ Multiplying the original differential equation by this function yields

$$x^{-2}y - x^{-1}y' = 0.$$

This equation is exact since

$$\begin{aligned}\frac{\partial}{\partial y}(x^{-2}y) &= x^{-2}\\ \frac{\partial}{\partial x}\left(-x^{-1}\right) &= x^{-2}\end{aligned}$$

so $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Now we look for a $\phi(x,y)$ such that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= x^{-2}y\\ \frac{\partial\phi}{\partial y} &= -x^{-1}\end{aligned}$$

Taking anti-partial derivatives of each of these equations yields

$$\begin{aligned}\phi(x,y) &= \int x^{-2}y\partial x = -x^{-1}y + H_1(y)\\ \phi(x,y) &= \int -x^{-1}\partial y = -x^{-1}y + H_2(x).\end{aligned}$$

So we must have $H_1(y) = 0$, $H_2(x) = 0$, and so $\phi(x, y) = x^{-1}y = y/x$ The original differential equation is thus equivalent to an algebraic relation of the form

$$\frac{y}{x} = C$$

or

$$y = Cx.$$

4. Solve the following first order differential equations using the substitution $u = y/x$.

(a) $xy' - y = \sqrt{xy}$

- This equation is equivalent to

(3) $$y' = \frac{1}{x}\left(\sqrt{xy} + y\right) = \sqrt{\frac{y}{x}} + \frac{y}{x} = F\left(\frac{y}{x}\right)$$

where

$$F(u) \equiv u^{1/2} + u.$$

Thus the equation is homogeneous of degree zero. If we define $u \equiv y/x$, we have $u' = y'/x - y/x^2$ and solving the latter equation for y' yields

$$y' = xu' + \frac{y}{x} = xu' + u.$$

If we now substitute $xu' + u$ for y' on the left hand side of (3) we obtain

$$xu' + u = y' = \sqrt{\frac{y}{x}} + \left(\frac{y}{x}\right) = u^{1/2} + u$$

or, cancelling the term u that appears on both sides,

$$xu' = u^{1/2}$$

or

$$u^{-1/2}du = \frac{dx}{x}.$$

Integrating both sides yields

$$2u^{1/2} = \ln|x| + C$$

or

$$2\sqrt{\frac{y}{x}} = \ln|x| + C.$$

Solving for y yields

$$y(x) = 4x\left(\ln|x| + C\right)^2$$

(b) $y' = \dfrac{y^2 + xy}{x^2}$, $y(1) = 1$

- This equation is equivalent to

$$y' = \frac{\frac{1}{x^2}}{\frac{1}{x^2}}\left(\frac{y^2 + xy}{x^2}\right) = \frac{\left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right)}{1} = \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right) = F\left(\frac{y}{x}\right)$$

where

$$F(u) \equiv u^2 + u.$$

Substituting $y = ux$, and $y' = xu' + u$, we obtain

$$xu' + u = y' = \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right) = u^2 + u$$

or

$$xu' = u^2$$

or

$$\frac{du}{u^2} = \frac{dx}{x}.$$

Integrating both sides yields

$$-\frac{1}{u} = \ln|x| + C$$

or

(4) $$-\frac{x}{y} = \ln|x| + C.$$

At this point it is convenient to employ the initial condition $y(1) = 1$ to fix the constant of integration C. Setting $x = 1$ and $y = 1$ in the equation above yields

$$-1 = \ln|1| + C = 0 + C = C.$$

So $C = 1$. Now we set $C = 1$ in (4) and solve for y to get

$$y(x) = \frac{-x}{\ln|x| - 1}.$$

(c) $3xyy' + x^2 + y^2 = 0$

- This equation is equivalent to

$$y' = \frac{-x^2 - y^2}{3xy} = -\frac{1}{3}\left(\frac{x}{y}\right) - \frac{1}{3}\left(\frac{y}{x}\right)$$

Making the substitutions $y = ux$, $y' = xu' + u$ yields

$$xu' + u = -\frac{1}{3}u^{-1} - \frac{1}{3}u$$

or

$$xu' = -\frac{1 + 4u^2}{u}$$

or

$$\frac{udu}{1 + 4u^2} = -\frac{1}{3}\frac{dx}{x}$$

Integrating both sides yields

$$\frac{1}{8}\ln\left(1 + 4u^2\right) = \int \frac{udu}{1 + 4u^2} = -\frac{1}{3}\ln|x| + C$$

or

$$\ln\left[1 + 4\left(\frac{y}{x}\right)^2\right] = C' + \ln|x^{-8/3}|$$

or

$$1 + 4\left(\frac{y}{x}\right)^2 = \exp\left[C' + \ln|x^{-8/3}|\right] = e^{C'}\exp\left[\ln|x^{-8/3}|\right] = C''x^{-8/3}$$

or, solving for y,

$$y(x) = \pm\frac{1}{2x}\sqrt{C''x^{-8/3} - 1}$$

5. Find a substitution that provides a solution to the following differential equations.

(a) $xy' + y = (xy)^3$

- In the hopes of simplifying the right hand side of the differential equation, we'll try $z = xy$. In this case, we'll have $y = z/x$ and $y' = z'/x - z/x^2$. Making the corresponding substitutions into the differential equation yields

$$x\left(\frac{1}{x}z' - \frac{z}{x^2}\right) + \frac{z}{x} = z^3$$

or

$$z' = z^3$$

or

$$\frac{dz}{z^3} = dx.$$

Integrating both sides yields

$$-\frac{1}{2}z^{-2} = x + C$$

or

$$z = \pm\frac{1}{\sqrt{(-2x - 2C)}}$$

Recalling that $z \equiv xy$, we obtain

$$yx = \pm\frac{1}{\sqrt{(-2x - 2C)}}$$

or

$$y(x) = \frac{\pm 1}{x\sqrt{(-2x - 2C)}}$$

(b) $(x + y)y' = (2x + 2y) \; - 3$

- For this equation it makes sense to try a substitution of the form $z = x + y$. In this case, we would have $y' = z' - 1$ and making
  the corresponding substitutions into the original differential equation would yield

$$z(z' - 1) = 2z - 3$$

or

$$z' = \frac{1}{z}(3z - 3) = \frac{3z - 3}{z}$$

or

$$\frac{zdz}{3z - 3} = dx.$$

Integrating both sides yields

$$= \frac{1}{3}z + \frac{1}{3}\ln(3z - 3) = \int \frac{zdz}{3z - 3} = \ln|x| + C$$

or

$$\frac{1}{3}(x + y) + \frac{1}{3}\ln(3(x + y) - 3) - \ln|x| = C$$

(Since we won't be able to solve this equation explicitly for y, the best we can do is express the solution as an algebraic equation.)