Math 2233
Homework Set 3

1. Determine the order of the following differential equations and whether or not the equations are linear.

(a) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + 2y = \sin(x)$

- This is second order linear ODE.

(b) $(1+y^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = e^x$

- This is s second order nonlinear ODE (the y^2 term makes it non-linear)..

(c) $\frac{d^4y}{dx^4} + \frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 1$

- This is a fourth order linear ODE.

(d) $\frac{dy}{dx} + xy^2 = 0$

- This is a first order non-linear ODE (the y^2 term makes it nonlinear).

(e) $\frac{d^2y}{dx^2} + \sin(x+y) = \sin(x)$

- This is a second order nonlinear ODE ($\sin(x+y)$ is a nonlinear function of y).

2. Solve $y' + 3y = x + e^{-2x}$.

- This equation is already in standard form with

$$\begin{aligned} p(x) &= 3 \\ g(x) &= x + e^{-2x} \end{aligned}$$

We can calculate the solution to this first order linear ODE using the formula

$$y(x) = \frac{1}{\mu(x)} \int^x \mu(s)g(s)ds + \frac{C}{\mu(x)}$$

where

$$\mu(x) = \exp\left[\int^x p(x)ds\right] .$$

First, we calculate $\mu(x)$:

$$\mu(x) = \exp\left[\int^x (3)ds\right] = \exp[3x] = e^{3x}$$

Then we calculate $y(x)$:

$$\begin{aligned} y(x) &= \frac{1}{e^{3x}} \int^x e^{3s}\left(s + e^{-2s}\right) ds + \frac{C}{e^{3x}} \\ &= e^{-3x} \int^x \left(se^{3s} + e^s\right) ds + Ce^{-3x} \\ &= e^{-3x}\left(\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + e^x\right) + Ce^{-3x} \\ &= \frac{1}{3}x - \frac{1}{9} + e^{-2x} + Ce^{-3x} \end{aligned}$$

Thus

$$y(x) = \frac{1}{3}x - \frac{1}{9} + e^{-2x} + Ce^{-3x}$$

3. Solve

$$y' - y = 2e^{x} \quad .$$

- This is a first order linear ODE with

$$p(x) = -1 \ \ , \ \ g(x) = 2e^{x} \ .$$

So

$$\mu(x) = \exp\left[\int^{x} p(s)ds\right] = \exp\left[\int^{x} -ds\right] = \exp\left[-x\right] = e^{-x}$$

and

$$\begin{aligned} y(x) &= \frac{1}{\mu(x)}\int^{x} \mu(s)g(s)ds + \frac{C}{\mu(x)} \\ &= \frac{1}{e^{-x}}\int^{x} e^{-s}\left(2e^{s}\right)ds + \frac{C}{e^{-x}} \\ &= e^{x}\int^{x} 2ds + Ce^{x} \\ &= 2xe^{x} + Ce^{x} \end{aligned}$$

Thus,

$$y(x) = 2xe^{x} + Ce^{x} \ .$$

4. Solve

$$xy' + 2y = \sin(x) \quad ; \quad x > 0 \quad .$$

- This is another first order linear ODE. However, before we can apply our formular, we must correctly identify the functions $p(x)$ and $g(x)$. Dividing through by x we cast the differential equation into standard form

$$y' + \frac{2}{x}y = \frac{1}{x}\sin(x).$$

Hence,

$$p(x) = \frac{2}{x} \ \ , \ \ g(x) = \frac{1}{x}\sin(x) \ .$$

Now we can calculate $\mu(x)$:

$$\begin{aligned} \mu(x) &= \exp\left[\int^{x} p(s)ds\right] \\ &= \exp\left[\int^{x} \frac{2}{s}ds\right] \\ &= \exp\left[2\ln|x|\right] \\ &= \exp\left[\ln|x^{2}|\right] \\ &= x^{2} \end{aligned}$$

And now that we have $\mu(x)$ we can calculate $y(x)$.

$$\begin{aligned} y(x) &= \frac{1}{\mu(x)} \int^x \mu(s) g(s) ds + \frac{C}{\mu(x)} \\ &= \frac{1}{x^2} \int^x s^2 \left(\frac{1}{s} \sin(s) \right) ds + \frac{C}{x^2} \\ &= \frac{1}{x^2} \int^x s \sin(s) ds + \frac{C}{x^2} \\ &= \frac{1}{x^2} \left(-x \cos(x) + \sin(x) \right) + \frac{C}{x^2} \end{aligned}$$

Hence,

$$y(x) = -\frac{1}{x} \cos(x) + \frac{1}{x^2} \sin(x) + \frac{C}{x^2} \; .$$

5. Solve the following initial value problem.

$$y' - y = 2xe^{2x} \quad ; \quad y(1) = 0 \; .$$

- This is a first order linear ODE with $p(x) = -1$ and $g(x) = 2xe^{2x}$. So

  $$\mu(x) = \exp\left[\int^x p(s) ds \right] = \exp\left[\int^x -ds \right] = \exp\left[-x \right] = e^{-x}$$

  hence, the general solution of the ODE is

  $$\begin{aligned} y(x) &= \frac{1}{\mu(x)} \int^x \mu(s) g(s) ds + \frac{C}{\mu(x)} \\ &= \frac{1}{e^{-x}} \int^x e^{-s} \left(2se^{2s} \right) ds + \frac{C}{e^{-x}} \\ &= e^x \int^x 2se^s ds + Ce^x \\ &= e^x \left(2xe^x - 2e^x \right) + Ce^x \\ &= 2xe^{2x} - 2e^{2x} + Ce^x \end{aligned}$$

  We now impose the initial condition $y(1) = 0$:

  $$\begin{aligned} 0 = y(1) &= 2(1)e^2 - 2e^2 + Ce^1 \\ &= Ce \end{aligned}$$

  Thus, $C = 0$ and so the solution to the initial value problem is

  $$y(x) = 2xe^{2x} - 2e^{2x}.$$

6. Solve the following initial value problem.

$$y' + \frac{2}{x} y = \frac{\cos(x)}{x^2} \quad ; \quad y(\pi) = 0$$

- This is a first order linear ODE with $p(x) = \frac{2}{x}$ and $g(x) = \frac{\cos(x)}{x^2}$. Hence

  $$\mu(x) = \exp\left[\int^x p(s) ds \right] = \exp\left[\int^x \frac{2}{s} ds \right] = \exp\left[2 \ln |x| \right] = \exp\left[\ln |x^2| \right] = x^2$$

and so the general solution of the ODE is

$$\begin{aligned} y(x) &= \frac{1}{\mu(x)}\int^x \mu(s)g(s)ds + \frac{C}{\mu(x)} \\ &= \frac{1}{x^2}\int^x s^2\left(\frac{\cos(s)}{s^2}\right)ds + \frac{C}{x^2} \\ &= \frac{1}{x^2}\int^x \cos(s)ds + \frac{C}{x^2} \\ &= \frac{1}{x^2}\sin(x) + \frac{C}{x^2} \end{aligned}$$

We now impose the initial condition to fix C.

$$0 = y(\pi) = \frac{1}{\pi^2}\sin(\pi) + \frac{C}{\pi^2} = 0 + \frac{C}{\pi^2} = \frac{C}{\pi^2}$$

So we must take $C = 0$. The solution to the initial value problem is thus

$$y(x) = \frac{\sin(x)}{x^2}.$$

7. Find the solution of the initial value problem below. State the interval in which the solution is valid.

$$xy' + 2y = x^2 - x + 1 \ ; \ y(1) = \frac{1}{2}.$$

- Dividing both sides by x we put the differential equation in standard form:

$$y' + \frac{2}{x}y = x - 1 + \frac{1}{x}$$

so $p(x) = \frac{2}{x}$ and $g(x) = x - 1 + \frac{1}{x}$. Note that since $p(x)$ and $g(x)$ are both undefined at $x = 0$, we might expect trouble for any solution we construct at the point $x = 0$. At any rate

$$\mu(x) = \exp\left[\int^x p(s)ds\right] = \exp\left[\int^x \frac{2}{s}ds\right] = \exp\left[2\ln|x|\right] = x^2$$

and so the general solution is

$$\begin{aligned} y(x) &= \frac{1}{\mu(x)}\int^x \mu(s)g(s)ds + \frac{C}{\mu(x)} \\ &= \frac{1}{x^2}\int^x s^2\left(s - 1 + \frac{1}{s}\right)ds + \frac{C}{x^2} \\ &= \frac{1}{x^2}\int \left(s^3 - s^2 + s\right)ds + \frac{C}{x^2} \\ &= \frac{1}{x^2}\left(\frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{2}x^2\right) + \frac{C}{x^2} \\ &= \frac{1}{4}x^2 - \frac{1}{3}x + \frac{1}{2} + \frac{C}{x^2} \end{aligned}$$

Note that if $C \neq 0$ then a solution is undefined at $x = 0$. Now we plug into the initial condition

$$\frac{1}{2} = y(1) = \frac{1}{4}(1)^2 - \frac{1}{3}(1) + \frac{1}{2} + \frac{C}{(1)^2} = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + C = \frac{5}{12} + C$$

so $C = \frac{1}{12}$.Thus the solution to the initial value problem is

$$y(x) = \frac{1}{4}x^2 - \frac{1}{3}x + \frac{1}{2} + \frac{1}{12}x^{-2}$$

which is which is well-defined on any interval that excludes the point $x = 0$.

8. Find the solution of the initial value problem below. State the interval in which the solution is valid.

$$y' + y = \frac{1}{1+x^2} \quad , \quad y(0) = 0 \quad .$$

- The differential equation is in standard form and the coefficient functions $p(x) = 1$ and $g(x) = \frac{1}{1+x^2}$ are well-defined for all x so we can expect solutions to be well defined on any subinterval of the real line. Calculating $\mu(x)$ we get

$$\mu(x) = \exp\left[\int^x p(s)ds\right] = \exp\left[\int^x ds\right] = e^x$$

and so the general solution will be

$$\begin{aligned} y(x) &= \frac{1}{\mu(x)}\int^x \mu(s)g(s)ds + \frac{C}{\mu(x)} \\ &= \frac{1}{e^x}\int^x \frac{e^s}{1+s^2}ds + \frac{C}{e^x} \\ &= e^{-x}\int^x \frac{e^s}{1+s^2}ds + \frac{C}{e^x} \end{aligned}$$

Unfortunately, there is no way to evaluate the integral

$$\int \frac{e^x}{1+x^2}dx$$

in closed form. To make further progress, we need to use the following formula for the solution of an initial value problem of the form $y' + p(x)y = g(x)$, $y(x_o) = y_o$.

$$y(x) = \frac{1}{\mu_o(x)}\int_{x_o}^x \mu_o(s)g(s)ds + \frac{y_o}{\mu_o(x)} \tag{1}$$

where

$$\mu_o(x) = \exp\left[\int_{x_o}^x p(s)ds\right] \tag{2}$$

Note the use of definite integrals in these formulas. In accordance with the initial condition $y(0) = 0$, we set $x_o = 0$ and $y_o = 0$; and
plug into the formulas (1) and (2):

$$\mu_o(x) = \exp\left[\int_0^x p(s)ds\right] = \exp\left[\int_0^x ds\right] = \exp[x - 0] = e^x$$

$$\begin{aligned} y(x) &= \frac{1}{\mu_o(x)}\int_{x_o}^x \mu_o(s)g(s)ds + \frac{y_o}{\mu_o(x)} \\ &= \frac{1}{e^x}\int_0^x \frac{e^s}{1+s^2}ds + \frac{0}{e^x} \\ &= \frac{1}{e^x}\int^x \frac{e^s}{1+s^2}ds \end{aligned}$$

9. The equation below has a discontinuity at $x = 0$. Solve the differential equation for $x > 0$ and describe the behavior of the solution as $x \to 0$ for various values of the constant of integration. Sketch several solutions. (You can use Maple to plot the direction fields and then sketch the graphs of several solutions on top of that plot by hand.)

$$y' + \tfrac{2}{x}y \quad = \quad \tfrac{1}{x^2} \quad .$$

- Let's first find the general solution of the differential equation. This is a first order linear ODE with $p(x) = \frac{2}{x}$ and $g(x) = \frac{1}{x^2}$, so

$$\mu(x) = \exp\left[\int^x p(s)ds\right] = \exp\left[\int^x \frac{2}{s}ds\right] = \exp\left[2\ln|x|\right] = x^2$$

$$\begin{aligned} y(x) &= \frac{1}{\mu(x)}\int^x \mu(s)g(s) + \frac{C}{\mu(x)} \\ &= \frac{1}{x^2}\int^x s^2\left(\frac{1}{s^2}\right)ds + \frac{C}{x^2} \\ &= \frac{1}{x^2}\int^x ds + \frac{C}{x^2} \\ &= \frac{1}{x^2}(x) + \frac{C}{x^2} \\ &= \frac{1}{x} + \frac{C}{x^2}. \end{aligned}$$

Thus, the general solution is

$$y(x) = \frac{1}{x} + \frac{C}{x^2}$$

These solutions (for various values of C) are all discontinuous at $x = 0$. Since, as x approaches 0, $\left|\frac{1}{x^2}\right|$ goes to infinity much faster than $\left|\frac{1}{x}\right|$, the behavior of these solutions as $x \to 0$ is controlled by the term $\frac{C}{x^2}$. It C is positive then $\lim_{x\to 0} y(x) = +\infty$, and if C is negative then $\lim_{x\to 0} y(x) = -\infty$ If $C = 0$ then $y(x) = \frac{1}{x}$ and so it goes to positive infinity when it approaches $x = 0$ from the right and to negative infinity when x approaches 0 from the left. Below are plots of $y(x)$ for $C = -2, -1, 0, 1, 2$.