Math 2233
Homework Set 2

1. Find the first four terms of the Taylor exansion about $x = 0$ of the solution of

$$y' = y^2 \tag{1}$$
$$y(0) = 1 \tag{2}$$

- The Taylor expansion of the solution $y(x)$ about $x = 0$ is

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} \frac{1}{n!} y^{(n)}(0) x^n \\ &= y(0) + y'(0)x + \frac{1}{2} y''(0)x^2 + \frac{1}{6} y'''(0)x^3 + \cdots \end{aligned}$$

To write down the first four terms of this expansion explicitly, we need to calculate $y(0)$, $y'(0)$, $y''(0)$, and $y'''(0)$. The initial condtion $y(0) = 1$ already gives us $y(0)$. To calculate $y'(0)$, we simply evaluate the differential equation at $x = 0$:

$$y'(0) = y^2\big|_{x=0} = (y(0))^2 = (1)^2 = 1. \tag{3}$$

To calculate $y''(0)$ we first differentiate the differential equation

$$y''(x) = \frac{d}{dx} y'(x) = \frac{d}{dx}\left(y^2(x)\right) = 2y(x)y'(x) \tag{4}$$

Evaluating this equation at $x = 0$ yields

$$y''(0) = 2y(0)y'(0) = 2(1)(1) = 2 \tag{5}$$

where we have used (2) and (3) to reduce the right hand side to a pure number. To calculated $y'''(0)$ we differentiate (4) and evaluate the right hand side at $x = 0$ using (2), (3), and (5):

$$\begin{aligned} y'''(0) &= \left.\frac{d}{dx}\left(2y(x)y'(x)\right)\right|_{x=0} \\ &= \left.(2y'(x)y'(x)) + 2y(x)y''(x)\right|_{x=0} \\ &= 2\left(y'(0)\right)^2 + 2y(0)y''(0) \\ &= 2(1)^2 + 2(1)(2) \\ &= 6 \end{aligned}$$

Hence,

$$\begin{aligned} y(x) &= y(0) + y'(0)x + \frac{1}{2} y''(0)x^2 + \frac{1}{6} y'''(0)x^3 + \cdots \\ &= 1 + x + \frac{1}{2}(2)x^2 + \frac{1}{6}(6)x^3 + \cdots \\ &= 1 + x + x^2 + x^3 + \cdots \end{aligned}$$

2. Find the first four terms of the Taylor expansion about $x = 1$ of the solution of

$$\begin{aligned} y' &= x^2 \\ y(1) &= 1 \end{aligned}$$

- In this problem we seek a Taylor expansion of our solution $y(x)$ about $x = 1$, and so we set

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n \\ &= y(1) + y'(1)(x-1) + \frac{1}{2} y''(1)(x-1)^2 + \frac{1}{6} y'''(1)(x-1)^3 + \cdots \end{aligned} \tag{6}$$

and try to calculate $y(1)$, $y'(1)$, $y''(1)$, and $y'''(1)$. The initial condition gives us

$$y(1) = 1. \tag{7}$$

Evaluating the differential equation at $x = 1$ gives us

$$y'(1) = x^2\big|_{x=1} = (1)^2 = 1. \tag{8}$$

Differentiating the differential equation and evaluating the result at $x = 1$ yields

$$y''(1) = \left.\frac{d}{dx}y'(x)\right|_{x=1} = \left.\frac{d}{dx}x^2\right|_{x=1} = 2x|_{x=1} = 2 \tag{9}$$

Finally, differentiating the differential equation twice and evaluating the result at $x = 1$ yields

$$y'''(1) = \left.\frac{d^2}{dx^2}y'(x)\right|_{x=1} = \left.\frac{d^2}{dx^2}x^2\right|_{x=1} = 2|_{x=1} = 2 \tag{10}$$

Plugging (7), (8), (9), and (10) into (6) yields

$$\begin{aligned} y(x) &= 1 + (1)(x-1) + \frac{1}{2}(2)(x-1)^2 + \frac{1}{6}(2)(x-1)^3 + \cdots \\ &= 1 + (x-1) + (x-1)^2 + \frac{1}{3}(x-1)^3 + \cdots \end{aligned}$$

3. Solve the following differential equation using Separation of Variables.

$$\frac{dy}{dx} = xe^y$$

- We can explicitly separate the x-dependence from the y-dependence in this equation by multiplying both sides by $e^{-y}dx$:

$$e^{-y}dx\left(\frac{dy}{dx} = xe^y\right) \quad \Rightarrow \quad e^{-y}dy = xdx$$

Integrating both sides of the resulting equation yields

$$-e^{-y} = \int e^{-y}dy = \int xdx = \frac{1}{2}x^2 + C$$

or

$$e^{-y} = C' - \frac{1}{2}x^2$$

or

$$-y = \ln\left|C' - \frac{1}{2}x^2\right|$$

or

$$y(x) = -\ln\left|C' - \frac{1}{2}x^2\right|.$$

4. Solve the following differential equation using Separation of Variables.

$$\frac{dx}{dt} = txe^{t^2}$$

- Multiplying both sides of this equation by $\frac{1}{x}dt$ yields

$$\frac{dx}{x} = te^{t^2}dt$$

and so the equation is separable. Integrating both sides we get

$$\ln|x| = \int\frac{dx}{x} = \int te^{t^2}dt = \int\frac{1}{2}e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{t^2} + C$$

where we have used the substitutions $u = t^2$, $du = 2tdt$, to carry out the integration over t. Solving the extreme sides of this equation for x yields

$$x(t) = \exp\left(\frac{1}{2}e^{t^2} + C\right) = \tilde{C}\exp\left(\frac{1}{2}e^{t^2}\right)$$

5. Solve the following differential equation using Separation of Variables.

$$x^2y' + e^y = 0$$

- Taking the e^y term to the right hand side and then multiplying by $x^{-2}e^{-y}dx$ yields

  $$e^{-y}dy = -\frac{dx}{x^2}$$

  Integrating both sides of this equation yields

  $$-e^{-y} = \int e^{-y}dy = \int -\frac{dx}{x^2} = -\left(-\frac{1}{x}\right) + C$$

  Solving this equation for y yields

  $$y(x) = \ln\left|C' - \frac{1}{x}\right|.$$

6. Solve the following differential equation using Separation of Variables.

$$yy' = e^x$$

- Multiplying both sides by dx yields

  $$ydy = e^x dx.$$

  Integrating both sides of this equation produces

  $$\frac{1}{2}y^2 = \int ydy = \int e^x dx = e^x + C$$

  Solving the extreme sides of this equation for y yields

  $$y(x) = \pm\sqrt{2e^x + C'}\ .$$