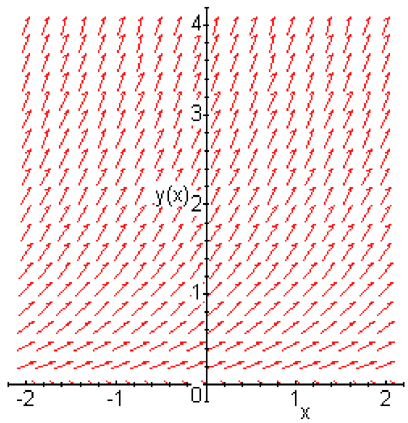
Math 2233
Homework Set 1

1.

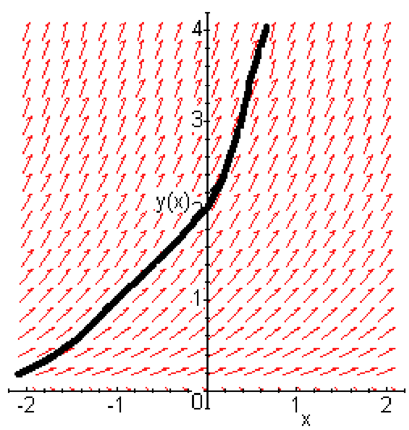
(a) Plot the direction field for the differential equation

$$y' = y^{4/5}.$$

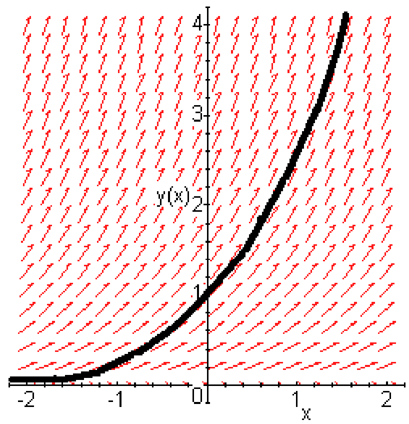
- Below is the Maple plot produced by the commands
  1. `with(DEtools);`
  2. `dfieldplot(diff(y(x)=y^(4/5),[y],x=0..4, y=-2..2);`



(b) Sketch the solution that satisfies $y(0) = 2$.



(c) Sketch the solution that satisfies $y(0) = 1$.

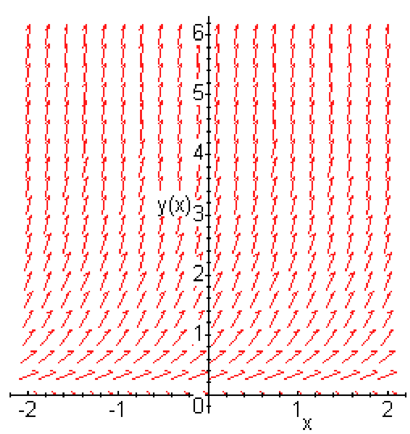


2. Use Maple to generate direction fields for the following differential equations on the given interval.

(a) $y' = 2y$; $-2 \leq x \leq 2$, $0 \leq y \leq 6$.

- The plot below was produced via the Maple commands

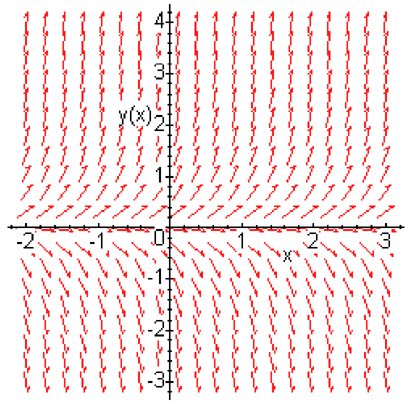
```
> with(DEtools);
> dfieldplot(diff(y(x)=2*y,[y],x=-2..2, y=0..6);
```



(b) $y' = 3y(1 - y)$; $-2 \leq x \leq 3$, $-3 \leq y \leq 4$.

- The plot below was produced using the Maple commands

```
> with(DEtools);
> dfieldplot(diff(y(x)=3*y*(1-y),[y],x=-2..3, y=-3..4);
```



3. For the differential equation in Problem 2(b), what can you say about the behavior of solutions as $x \to \infty$?

- By virtue of the differential equation

$$y' = 3y(1-y)$$

we see that the value of y determines whether a solution $y(x)$ is increasing, decreasing, or constant (that is to say, when $y'(x)$ is positive, negative, or zero).< 0 are decreasing.
1. If $y > 1$, then $y' < 0$, and so solutions in the region $y > 1$ are always decreasing.
2. if $y = 1$, then $y' = 0$, and so solutions for which $y = 1$ are constant and so stick to the line $y = 1$.
3. if $0 < y < 1$, then $y' > 0$, and so solutions in the region $0 < y < 1$ are increasing.
4. if $y = 0$, then $y' = 0$, and so solutions for which $y = 0$ are constant, and so stick to the line $y = 0$.
5. if $y < 0$, then $y' < 0$, and so solutions in the region y

We thus have four basic classes of solutions. The solutions in the region $y > 1$ are always decreasing, but they cannot pass through the line $y = 1$, since that corresponds to a constant solution. These solutions asymptotically approach the line $y = 1$ as $x \to \infty$. The solutions in the region $0 < y < 1$ are always increasing, but they cannot increase past the line $y = 1$ because again $y(x) = 1$ is a constant solution. These solutions must also asymptotically approach the line $y = 1$ (from below). The solutions in the region $y < 0$ are always decreasing, these solutions must tend to $-\infty$ as $x \to \infty$.

4. Using the Euler Method, find an approximate value for $y(1)$ for the following initial value problem (take $h = \Delta x = 0.02$):

$$\frac{dy}{dx} = x + y \quad , \quad y(0) = 1$$

- We'll do this problem by hand. In accordance with the initial condition $y(0) = 1$ we set $x_0 = 0$ and $y_0 = 1$. To get the next pair of points on the solution curve we use the fact that the slope of the best straight line fit to the solution curve at $(x_0, y_0) = (0, 1)$ must be

$$m_0 = \left.\frac{dy}{dx}\right|_{(x_0,y_0)} = x_0 + y_0 = 0 + 1 = 1.$$

Setting

$$x_1 = x_0 + \Delta x = 0 + .2 = 0.2$$

we get an approximate value for $y_1 = y(x_1)$ using the formula $\Delta y = m\Delta x$; (for the case at hand, this formula implies $y_1 = y_0 + m_0\Delta x$)

$$\begin{aligned} y_1 &= y_0 + m_0\Delta x \\ &= 1 + (1)(0.2) \\ &= 1.2 \end{aligned}$$

Thus the next pair of points on the solution curve should be $(x_1, y_1) = (0.2, 1.2)$. Now we calculate the slope of the best strainght line fit the to solution that passes through the point (x_1, y_1):

$$m_1 = \left.\frac{dy}{dx}\right|_{(x_1,y_1)} = x_1 + y_1 = 0.2 + 1.2 = 1.4$$

Taking $x_2 = x_1 + \Delta x = 0.4$, we calculate y_2

$$\begin{aligned} y_2 &= y_1 + m_1\Delta x \\ &= 1.2 + (1.4)(0.2) \\ &= 1.48 \end{aligned}$$

We continue in this manner:

$$\begin{aligned} m_2 &= x_2 + y_2 = 0.4 + 1.48 = 1.88 \\ x_3 &= x_2 + \Delta x = 0.4 + 0.2 = 0.6 \\ y_3 &= y_2 + m_2\Delta x = 1.48 + (1.88)(0.2) = 1.856 \\ m_3 &= x_3 + y_3 = 0.6 + 1.856 = 2.456 \\ x_4 &= x_3 + \Delta x = 0.6 + 0.2 = 0.8 \\ y_4 &= y_3 + m_3\Delta x = 1.856 + (2.456)(0.2) = 2.3472 \\ m_4 &= x_4 + y_4 = 0.8 + 2.3472 = 3.1472 \\ x_5 &= x_4 + \Delta x = 0.8 + 0.2 \\ y_5 &= y_4 + m_4\Delta x = 2.3472 + (3.1472)(0.2) = 2.9766 \end{aligned}$$

Thus $y(1) = y(x_5) = y_5 = 2.9766$.

5. Using the Euler Method, find an approximate value for $y(1)$ for the following initial value problem (take $h = \Delta x = 0.1$):

$$\frac{dy}{dx} = xe^y \quad , \quad y(0) = 0$$

- For this problem we'll resort to Maple. The algorithm used in the previous problem can generalzed and applied to this problem as follows.
  (i) In accordance with the initial condition $y(0) = 0$ set $x_0 = 0$ and $y_0 = 0$.
  (ii) Set $\Delta x = 0.1$.
  (iii) We divide the interval $[0, 1]$ into $10 = \frac{1-0}{\Delta x}$ subintervals by setting

  $$x_n = 0 + n\Delta x \quad n = 0, 1, \ldots, 10$$

  (iv) We set

  $$y_n = y(x_n) \ , \ n = 0.1, \ldots, 10$$

  Our goal is to calculate $y_{10} = y(x_{10}) = y(1)$.
  (v) The slope of the solution passing through the point (x_i, y_i) is determined by the right hand side of the differential equation evaluated at (x_i, y_i):

  $$m_i = \left.\frac{dy}{dx}\right|_{(x_i,y_i)} = \left. xe^y\right|_{(x_i,y_i)} = x_i e^{y_i}$$

(vi) Given a point (x_i, y_i) on the solution curve we can approximate y_{i+1} by using the formula

$$m_i = \frac{\Delta y}{\Delta x} = \frac{y_{i+1} - y_i}{\Delta x}$$

or, after solving for y_{i+1} is equivalent to

$$\begin{aligned} y_{i+1} &= y_i + m_i \Delta x \\ &= y_i + (x_i e^{y_i}) \Delta x \end{aligned}$$

Thus, y_{i+1} is completely determined by the preceding values x_i and y_i of, respectively, x and y.

(vii) Now we can explicitly state the algorithm by which we will calculate $y(1) = y_{10}$.
We set $x_0 = 0$, $y_0 = 0$ and $\Delta x = 0.1$. For each i from 0 to 9 we will successively calculate

$$\begin{aligned} x_{i+1} &= 0 + (i+1)\Delta x \\ y_{i+1} &= y_i + (x_i e^{y_i}) \Delta x. \end{aligned}$$

Below is the Maple routine that accomplishes this.

```
> x[0] := 0;
> y[0] := 0;
> dx := 0.1;
> for i from 0 to 9 do
> x[i+1] := (i+1)*dx:
> y[i+1] := y[i]+ x[i]*exp(y[i]):
> od:
> y[10];
```

The result calculated by Maple is $y_{10} = .5653922980$.