Math 2233.004
SECOND EXAM
12:30 - 1:45 pm, April 8,1999

1. Given that $y_1(x) = x^2$ and $y_2(x) = x^{-3}$ are solutions to $x^2y'' + 2xy' - 6y = 0$.

(a) (5 pts) Show that the functions $y_1(x)$ and $y_2(x)$ are linearly independent.

- $$W[y_1, y_2] = y_1y_2' - y_1'y_2 = \left(x^2\right)\left(-3x^{-4}\right) - (2x)\left(x^{-3}\right) = -5x^{-2} \neq 0$$

  So y_1 and y_2 are linearly independent. $\square$

(b) (5 pts) Write down the general solution of $x^2y'' + 2xy' - 6y = 0$.

- y_1 and y_2 are two linearly independent solutions of a 2^{nd} order homogeneous linear ODE, so the general solution will be

$$\begin{aligned} y(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1x^2 + c_2x^{-3} \end{aligned}$$

$\square$

(c) (5 pts) Find the solution satisfying the initial conditions $y(1) = 1$, $y'(1) = 0$.

- $$\begin{aligned} 1 &= y(1) = c_1y_1(1) + c_2y_2(1) = c_1 + c_2 \\ 0 &= y'(1) = c_1y_1'(1) + c_2y_2'(1) = 2c_1 - 3c_2 \end{aligned}$$

  Solving for c_1 and c_2 yields

$$\begin{array}{l} c_1 = \frac{3}{5} \\ c_2 = \frac{2}{5} \end{array} \quad \Rightarrow \quad y(x) = \frac{3}{5}x^2 + \frac{2}{5}x^{-3}$$

$\square$

2. (10 pts) Given that $y_1(x) = x^{-2}$ is one solution of $x^2y''+5xy'+4y = 0$, use Reduction of Order(explicitly) to determine the general solution.

- $$\begin{aligned} y_2(x) &= y_1(x)\int^x \frac{1}{\left[y_1(x_1)\right]^2}\exp\left[-\int^{x_1} p(x_2)dx_2\right]dx_1 \\ &= x^{-2}\int^x \frac{1}{(x_1)^{-4}}\exp\left[-\int^{x_1}\frac{5}{x_2}dx_2\right]dx_1 \\ &= x^{-2}\int^x (x_1)^4\left(x_1^{-5}\right)dx_1 \\ &= x^{-2}\ln|x| \end{aligned}$$

$$y(x) = c_1y_1(x) + c_2y_2(x) = c_1x^{-2} + c_2x^{-2}\ln|x|$$

$\square$

3. Given that $y_1(x) = x$ and $y_2(x) = x^2$ are solutions of $x^2y'' - 2xy' + 2y = 0$.

(a) (10 pts) Use the Method of Variation of Parameters to find the general solution of
$$x^2y'' - 2xy' + 2y = x \quad .$$

- $$\begin{aligned}
W[y_1, y_2] &= (x)(2x) - (1)(x^2) = x^2 \\
p(x) &= -\frac{2}{x} \\
g(x) &= \frac{1}{x} \\
y_p(x) &= -y_1 \int \frac{y_2 g}{W[y_1, y_2]} dx + y_2 \int \frac{y_1 g}{W[y_1, y_2]} dx \\
&= -x \int \frac{(x^2)(x^{-1})}{x^2} dx + x^2 \int \frac{(x)(x^{-1})}{x^2} dx \\
&= -x (\ln|x|) + x^2 \left(-\frac{1}{x}\right) \\
&= -x \ln|x| - x
\end{aligned}$$

  The gereral solution on this 2^{nd} order, non-homogeneous, linear ODE is thus
$$\begin{aligned}
y(x) &= y_p(x) + c_1 y_1(x) + c_2 y_2(x) \\
&= -x \ln|x| - x + c_1 x + c_2 x^2 \\
&\approx -x \ln|x| + c_1 x + c_2 x^2
\end{aligned}$$

$\square$

(b) (5 pts) Find the solution satisfying $y(1) = 0$, $y'(1) = 2$.

- $$\begin{array}{l} 0 = y(1) = 0 + c_1 + c_2 \\ 2 = y'(1) = -1 + c_1 + 2c_2 \end{array} \quad \Rightarrow \quad \begin{array}{l} c_1 = -3 \\ c_2 = 3 \end{array} \quad \Rightarrow \quad y(x) = -x \ln|x| - 3x + 3x^2$$

$\square$

4. (10 pts) Given that $y_1(x) = x + 1$ is one solution of $y'' - (1 + x)y' + y = 0$, explain how one can construct a formula for the general solution of $y'' - (1 + x)y' + y = \sin(x)$. (You need not carry out any explicit computations.)

- (1) Given $y_1(x)$, use Reduction of Order formula to find a second linear independent solution $y_2(x)$ of $y'' - (1 + x)y' + y = 0$.

  (2) Now that you have two linear independent solutions of $y'' - (1 + x)y' + y = 0$, use Variation of Parameters formula to find a particular solution $y_p(x)$ of $y'' - (1 + x)y' + y = \sin(x)$.

  (3) The general solution of $y'' - (1 + x)y' + y = \sin(x)$ will be
$$y_p(x) = y_p(x) + c_1 y_1(x) + c_2 y_2(x)$$

$\square$

5. Determine the general solution of the following differential equations.

(a) (5 pts) $y'' - 6y' + 13y = 0$

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$$\begin{aligned} \lambda^2 - 6\lambda + 13 &= 0 \quad \Rightarrow \quad \lambda = \frac{6 \pm \sqrt{36-52}}{2} = 3 \pm 2i \\ y(x) &= c_1 e^{3x} \cos(2x) + c_2 e^{3x} \sin(2x) \end{aligned}$$

$\square$

(b) (5 pts) $y'' - 6y' + y = 0$

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$$\begin{aligned} \lambda^2 - 6\lambda + 1 &= 0 \quad \Rightarrow \quad \lambda = \frac{6 \pm \sqrt{36-4}}{2} = 3 \pm 2\sqrt{2} \\ y(x) &= c_1 e^{(3+2\sqrt{2})x} + c_2 e^{(3-2\sqrt{2})x} \end{aligned}$$

$\square$

(c) (5 pts) $y'' - 4y' + 4y = 0$

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$$\begin{aligned} \lambda^2 - 4\lambda + 4 &= 0 \quad \Rightarrow \quad \lambda = 2 \text{ (double root)} \\ y(x) &= c_1 e^{2x} + c_2 x e^{2x} \end{aligned}$$

$\square$

6. Find the general solution of the following Euler-type differential equations.

(a) (5 pts) $x^2 y'' - 4xy' - 6 = 0$

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$$\begin{aligned} 0 &= r(r-1) - 4r - 6 = r^2 - 5r - 6 = (r-3)(r-2) \quad \Rightarrow \quad r = 2, 3 \\ y(x) &= c_1 x^2 + c_2 x^3 \end{aligned}$$

$\square$

(b) (5 pts) $4x^2 y'' + y = 0$

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$$\begin{aligned} 0 &= 4r(r-1) + 1 = 4r^2 - 4r + 1 = (2r-1)(2r-1) \quad \Rightarrow \quad r = \frac{1}{2} \text{ (double root)} \\ y(x) &= c_1 x^{\frac{1}{2}} + c_2 x^{\frac{1}{2}} \ln|x| \end{aligned}$$

$\square$

(c) (5 pts) $x^2 y'' - 7xy' + 25y = 0$

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$$\begin{aligned} 0 &= r(r-1) - 7r + 25 = r^2 - 8r + 25 \quad \Rightarrow \quad r = \frac{8 \pm \sqrt{64-100}}{2} = 4 \pm 3i \\ y(x) &= c_1 x^4 \cos(3\ln|x|) + c_2 x^4 \sin(3\ln|x|) \end{aligned}$$

$\square$

7. Determine the general solution of the following differential equations.

(a) (10 pts) $y''' + y'' - 5y' + 3y = 0$

- $$\begin{aligned} 0 &= \lambda^3 + \lambda^2 - 5\lambda + 3 = (\lambda+3)\left(\lambda^2 - 2\lambda + 1\right) = (\lambda+3)(\lambda-1)^2 \\ y(x) &= c_1 e^{-3x} + c_2 e^{x} + c_3 x e^{3x} \end{aligned}$$

$\square$

(b) (10 pts) $y'''' + 8y'' + 16y = 0$

- $$\begin{aligned} 0 &= \lambda^4 + 8\lambda^2 + 16 = \left(\lambda^2 + 4\right)^2 = ((\lambda + 2i)(\lambda - 2i))^2 = (\lambda+2i)^2(\lambda-2i)^2 \\ y(x) &= c_1 e^0 \cos(2x) + c_2 x e^0 \cos(2x) + c_3 e^0 \sin(2x) + c_4 e^0 x \sin(2x) \\ &= c_1 \cos(2x) + c_2 x \cos(2x) + c_3 \sin(2x) + c_4 x \sin(2x) \end{aligned}$$

$\square$