Math 2233.04

SOLUTIONS TO FIRST EXAM

1. (10 pts.) Classify the following differential equations (identify their order, and state whether they are linear or non-linear, partial or ordinary, differential eqations).

(a) $\frac{dy}{dx} - x^2 y = \sin(x)$

- 1st order, linear, ordinary

(b) $y\frac{\partial^2\Phi}{\partial t^2} - \frac{\partial\Phi}{\partial y} = \Phi^2$

- 2nd order, non-linear, partial

(c) $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \sin(xy)$

- 2nd order, non-linear, ordinary

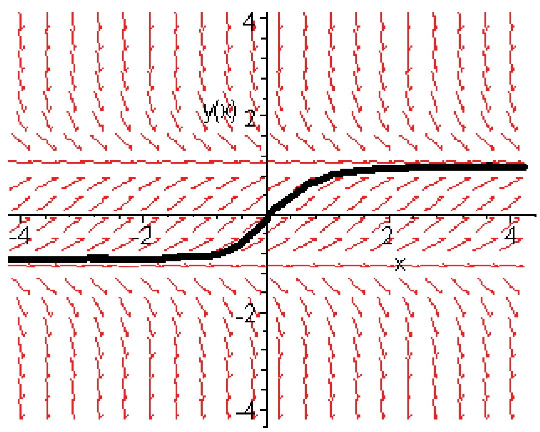
(d) $y^2\frac{\partial\Phi}{\partial x} + x^2\frac{\partial\Phi}{\partial x} = x$

- 1st order, linear, partial

(e) $\frac{d^3f}{dx^3} + x\frac{d^2f}{dx^2} + (2x+1)\frac{df}{dx} = f^2$

- 3rd order, non-linear, ordinary

2. Consider the plot below of the direction field for the differential equation $y' = -(y-1)(y+1)$.



(a) (5 pts) Sketch the solution curve satisfying $y(0) = 0$.

- The solution curve is superimposed on the figure above.

(b) (5 pts) Suppose $y(x)$ is a solution satisfying $y(0) = 2$. What can you say about the asymptotic behavior of $y(x)$ as $x \to \infty$? Explain.

- The derivative

$$y' = -(y-1)(y+1)$$

is always negative so long as $y > 1$, and is equal to zero when $y = 1$. So a solution $y(x)$ will be decreasing whenever $y > 1$, but its graph will flatten out as it approaches the line $y = 1$. Therefore,

$$\lim_{x\to\infty} y(x) = 1$$

for the solution that starts at $y(0) = 2$.

3. (10 pts.) Use a two-step numerical (Euler) method to find an approximate value for $y(1.2)$ where y is the solution of

$$\begin{aligned} \frac{dy}{dx} &= x + y^2 \\ y(1) &= 1 \end{aligned}$$

(Use a step size of $\Delta x = 0.1$).

- By virtue of the differential equation, the slope m of a solution curve passing through the point (x, y) must be $x + y^2$

$$\begin{aligned} x_0 &= 1 \\ y_0 &= 1 \\ x_1 &= x_0 + \Delta x = 1 + 0.1 = 1.1 \\ y_1 &= y_1 + m(x_0, y_0)\,\Delta x = 1 + (1 + 1^2)(0.1) = 1.2 \\ x_2 &= x_1 + \Delta x = 1.1 + 0.1 = 1.2 \\ y_2 &= y_1 + m(x_1, y_1)\,\Delta x = 1.2 + \left(1.1 + (1.2)^2\right)(0.1) = 1,454 \end{aligned}$$

So $y_2 \approx y(1.2) = 1.454$

4. (10 pts) Find the first three terms of the Taylor expansion (i.e., the terms to order $(x-1)^2$) of the solution of

$$\begin{aligned} \frac{df}{dx} &= xf^2 \\ f(1) &= 1 \end{aligned}$$

- We have

$$\begin{aligned} f(1) &= 1 \\ f'(1) &= \left.\frac{df}{dx}\right|_{x=1} = \left.\left[x\left(f(x)\right)^2\right]\right|_{x=1} = (1)(1)^2 = 1 \\ f''(1) &= \left.\frac{d^2f}{dx^2}\right|_{x=1} = \left.\frac{d}{dx}\left[x\left(f(x)\right)^2\right]\right|_{x=1} = \left.\frac{d}{dx}\left[\left(f(x)^2 + x2f(x)f'(x)\right)\right]\right|_{x=1} = 1^2 + (1)(2)(1)(1) = 3 \end{aligned}$$

So

$$\begin{aligned} f(x) &= f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 + \cdots \\ &= 1 + (x-1) + \frac{3}{2}(x-1)^2 + \cdots \end{aligned}$$

5. (10 pts) Solve the following initial value problem.

$$xy' - 2y = x^2 \qquad , \qquad y(1) = 3 \quad .$$

- This is a 1st order linear equation equivalent to

$$y' - \frac{2}{x}y = x \quad \Rightarrow \quad p(x) = -\frac{2}{x} \quad , \quad g(x) = x$$

So we can compute the general solution via the following formula

$$\begin{aligned} \mu(x) &= \exp\left[\int p(x)dx\right] = \exp\left[\int -\frac{2}{x}dx\right] = \exp\left[-2\ln|x|\right] = \exp\left[\ln\left[x^{-2}\right]\right] = x^{-2} \\ y(x) &= \frac{1}{\mu(x)}\int^x \mu(\tilde{x})g(\tilde{x})d\tilde{x} + \frac{C}{\mu(x)} = x^2\int^x \tilde{x}^{-2}(\tilde{x})\,d\tilde{x} + Cx^2 = x^2\int^x \frac{d\tilde{x}}{\tilde{x}} + Cx^2 = x^2\ln|x| + Cx^2 \end{aligned}$$

To fix the constant C we impose the initial condition

$$3 = y(1) = (1)^2 \ln|1| + C(1)^2 = 0 + C \qquad \Rightarrow \quad C = 3$$

Thus

$$y(x) = x^2 \ln|x| + 3x^2$$

6. (10 pts) Find the (explicit) solution of the following initial value problem. (Hint: the differential equation is separable.)

$$3x^2 - 2yy' = 1 \qquad , \qquad y(0) = 1 \qquad .$$

- We can rewrite this equation as

$$2y\frac{dy}{dx} = 3x^2 - 1 \quad \Rightarrow \quad 2y\,dy = \left(3x^2 - 1\right) dx$$

Integrating both sides yields

$$y^2 = \int 2y\,dy = \int \left(3x^2 - 1\right) dx + C = x^3 - x + C$$

We can fix the constant C by demanding that the initial point $(x, y) = (0, 1)$ lies on the solution curve

$$\Rightarrow \quad (1)^2 = y^2 = x^3 - x + C = (0)^3 - 0 + C \qquad \Rightarrow \quad C = 1$$

So

$$y^2 = x^3 - x + 1 \quad \Rightarrow \quad y = \pm\sqrt{x^3 - x + 1}$$

However, only the positive root will satisfy $y(0) = 1$, so

$$y(x) = \sqrt{x^3 - x + 1}$$

7. Consider the following differential equation.

$$2x + \sin(y) + (y + x\cos(y))\frac{dy}{dx} = 0$$

(a) (5 pts) Show that this equation is exact

- $$\left.\begin{array}{lll} M = 2x + \sin(y) & \Rightarrow & \frac{\partial M}{\partial y} = \cos(y) \\ N = y + x\cos(y) & \Rightarrow & \frac{\partial N}{\partial x} = \cos(y) \end{array}\right\} \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \Rightarrow \quad exact$$

(b) (10 pts) Find an implicit solution for this differential equation.

- $$\begin{aligned} \Psi(x,y) &= \int M\partial x + H_1(y) = \int (2x + \sin(y))\,\partial x + H_1(y) = x^2 + x\sin(y) + H_1(y) \\ \Psi(x,y) &= \int N\partial y + H_2(x) = \int (y + x\cos(y))\,\partial y + H_2(x) = \frac{1}{2}y^2 + x\sin(y) + H_2(x) \end{aligned}$$

Comparing these two expressions for $\Psi(x, y)$ we conclude that

$$\Psi(x,y) = x^2 + x\sin(y) + \frac{1}{2}y^2$$

and so the original differential equation is equivalent to the following algebraic equation (the implicit solution)

$$x^2 + x\sin(y) + \frac{1}{2}y^2 = C$$

8. (10 pts) Use a change of variable to solve the following differential equation.

$$\frac{dy}{dx} = \frac{xy - y^2}{x^2}$$

(Hint: this equation is homogeneous of degree 0.)

- We have

$$\frac{dy}{dx} = \frac{xy}{x^2} - \frac{y^2}{x^2} = \left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^2 = F\left(\frac{y}{x}\right) \quad \text{if} \quad F(z) \equiv z - z^2$$

So the equation is homogeneous of degree 0. Therefore, we can make the substitution on the right hand side and the substitution

$$y' = \frac{d}{dx}(xz) = z + xz'$$

on the left hand side. We thus obtain the following equivalent differential equation

$$z + xz' = F(z) = z - z^2 \quad \Rightarrow \quad x\frac{dz}{dx} = -z^2 \quad \Rightarrow \quad -\frac{dz}{z^2} = \frac{dx}{x}$$

Integrating both sides of this last equation (which is obviously separable and separated), we obtain

$$\frac{1}{z} = \ln|x| + C \qquad \Rightarrow \quad z = \frac{1}{\ln|x| + C} \qquad \Rightarrow \quad \frac{y}{x} = \frac{1}{\ln|x| + C}$$

or

$$y(x) = \frac{x}{\ln|x| + C}$$

9.

(a) (5 pts) Show that the following differential equation is not exact.

$$(1 - y^2)dx + (1 + x - y - xy)dy = 0 \quad .$$

$$\left.\begin{array}{lll} M = 1 - y^2 & \Rightarrow & \frac{\partial M}{\partial y} = -2y \\ N = 1 + x - y - xy & \Rightarrow & \frac{\partial N}{\partial x} = 1 - y \end{array}\right\} \quad \Rightarrow \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad \Rightarrow \quad \text{not exact}$$

(b) (10 pts) Find an integrating factor. (Hint: look for an integrating factor that depends only on y.)

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$$\begin{aligned} F_2 &\equiv \frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{1}{1 - y^2}\left[(1 - y) - (-2y)\right] \\ &= \frac{1 + y}{1 - y^2} = \frac{1 + y}{(1 - y)(1 + y)} = \frac{1}{1 - y} \end{aligned}$$

Since this depends only on y,

$$\begin{aligned} \mu(y) &= \exp\left[\int F_2(y)dy\right] = \exp\left[\int \frac{1}{1 - y}dy\right] \\ &= \exp\left[-\ln\left[1 - y\right]\right] = \exp\left[\ln\left[\frac{1}{1 - y}\right]\right] = \frac{1}{1 - y} \end{aligned}$$

should be an integrating factor. Sure enough

$$\begin{aligned} 0 &= \frac{1}{1 - y}\left[(1 - y^2)dx + (1 + x - y - xy)dy\right] \\ &= \frac{1}{1 - y}\left[(1 - y)(1 + y)dx + (1 + x)(1 - y)dy\right] \\ &= (1 + y)dx + (1 + x)dy \end{aligned}$$

is exact since

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(1+y) = 1 = \frac{\partial}{\partial x}\,(1+x) = \frac{\partial N}{\partial x}$$