LECTURE 27

Series Solutions about Regular Singular Points

Let's now consider the differential equation

$$2x^2y'' - xy' + (1+x)y = 0 \quad . \tag{27.1}$$

This equation evidently has a regular singular point at $x = 0$. We will look for a solution around $x = 0$ by making an ansatz for $y(x)$ by combining our ansatz for power series solutions about regular points with the ansatz we made for Euler type equations. More explicitly, we shall take

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} \quad . \tag{27.2}$$

We can suppose without loss of generality that $a_0 \neq 0$; i.e., we assume r to be chosen such that the first nonzero term in the series is $a_o x^r$. Plugging (27.2) into (27.1) yields

(27.3)
$$\begin{aligned}
0 &= 2x^2 \textstyle\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n-2} - x\sum_{n=0}^{\infty}(r+n)a_n x^{r+n-1} + (1+x)\sum_{n=0}^{\infty} a_n x^{r+n} \\
&= \textstyle\sum_{n=0}^{\infty} 2(r+n)(r+n-1)a_n x^{r+n} - \sum_{n=0}^{\infty}(r+n)a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+1} \\
&= \textstyle\sum_{n=0}^{\infty} \left(2(r+n)(r+n-1) - (r+n) + 1\right) a_n x^{r+n} + \sum_{n=1}^{\infty} a_{n-1} x^{r+n} \\
&= (2r)(r-1) - r + 1)\, a_0 + \textstyle\sum_{n=1}^{\infty} \left(\left(2(r+n)(r+n-1) - (r+n) + 1\right) a_n + a_{n-1}\right) x^{r+n}
\end{aligned}$$

Hence, we need

$$0 = (2r)(r-1) - r + 1 = 2r^2 - 3r + 1 \tag{27.4}$$

$$0 = a_{n-1} + \left(2(r+n)(r+n-1) - (r+n) + 1\right) a_n \tag{27.5}$$

The first relation is a quadratic equation for r. It is called the **indicial equation** for (27.1). Since

$$2r^2 - 3r + 1 = (2r-1)(r-1) \tag{27.6}$$

we must have

$$r = \frac{1}{2}, 1 \tag{27.7}$$

The second equation (27.5) furnishes a recursion relation that allows us to fix all coefficients a_n in terms of a_o and r.

Setting $r = \frac{1}{2}$ we have

$$\begin{aligned}
0 &= a_{n-1} + \left(2(\tfrac{1}{2}+n)^2 - 3(\tfrac{1}{2}+n) + 1\right) a_n \\
&= a_{n-1} + \left[n(2n-1)\right] a_n
\end{aligned} \tag{27.8}$$

so

$$a_n = \frac{-a_{n-1}}{n(2n-1)} \tag{27.9}$$

Thus,

$$\begin{aligned} a_1 &= \tfrac{-a_0}{(1)(2-1)} = -a_0 \\ a_2 &= \tfrac{-a_1}{(2)(4-1)} = \tfrac{a_0}{6} \\ a_3 &= \tfrac{-a_2}{(3)(6-1)} = \tfrac{-a_0}{90} \end{aligned} \tag{27.10}$$

So one solution would be

$$y_1(x) = a_0 x^{1/2}\left(1 - x + \frac{1}{6}x^2 - \frac{1}{90}x^3 + \cdots\right) \quad . \tag{27.11}$$

When $r = 1$ we have

$$0 = a_{n-1} + \left(2(1+n)^2 - 3(1+n) + 1\right) a_n \tag{27.12}$$

or

$$a_n = \frac{-1}{2(1+n)^2 - 3(1+n) + 1} a_{n-1} = \frac{-a_{n-1}}{n(2n+1)} \quad . \tag{27.13}$$

So

$$\begin{aligned} a_1 &= \tfrac{-a_0}{1(2+1)} = -\tfrac{a_0}{3} \\ a_2 &= \tfrac{-a_1}{2(4+1)} = \tfrac{a_0}{30} \\ a_3 &= \tfrac{-a_2}{3(6+1)} = -\tfrac{a_0}{630} \end{aligned} \tag{27.14}$$

Thus, a second solution of (27.1) would be

$$y_2(x) = a_o x\left(1 - \frac{1}{3}x + \frac{1}{30}x^2 - \frac{1}{630}x^3 + \cdots\right) \quad . \tag{27.15}$$

The general solution of (27.1) will be a linear combination of $y_1(x)$ and $y_2(x)$:

$$y(x) = c_1 x^{1/2}\left(1 - x + \frac{1}{6}x^2 - \frac{1}{90}x^3 + \cdots\right) + c_2 x\left(1 - \frac{1}{3}x + \frac{1}{30}x^2 - \frac{1}{630}x^3 + \cdots\right) \quad . \tag{27.16}$$

In summary, to find a solution of (27.1), we

1. Assume there is a solution of the form $y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$, with $a_0 \neq 0$.
2. Plug this expression for $y(x)$ into the differential equation and set the total coefficients of each power of x equal to zero. This lead to
   (i) a quadratic equation for r (the indicial equation)
   (ii) a set of recursion relations relating the coefficients a_n
3. Find the two roots r_1 and r_2 of the indicial equations, and then, for each root r_i used the recursion relations to express all the coefficients a_n in terms of a_o.
4. Write down a corresponding solution for each root y_i(x) for each root r_i of the indicial equation.
5. Write down the general solution as

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \quad .$$

WARNING: This technique works produces two linearly independent solutions only when:

(i) **There are two distinct roots $\mathbf{r}_1$ and $\mathbf{r}_2$ of the indicial equation.**
(ii) **The difference $\mathbf{r}_1 - \mathbf{r}_2$ is not an integer.**

See Sections 5.7 and 5.8 of the text for a discussion of what happens and how to procede when these criteria are not meet.