

Differential Equations with Polynomial Coefficients

In the last lecture we considered a number of examples of differential equations of the form

$$(25.1) \quad P(x)y'' + Q(x)y' + R(x)y = 0$$

and looked for solutions of the form

$$(25.2) \quad y(x) = \sum_{n=0}^{\infty} a_n (x - x_o)^n \quad .$$

Before considering one more example, let me first articulate the general procedure.

Step 1. Substitute (25.2) into (25.1). This will produce an equation of the form

$$(25.3) \quad 0 = \sum_{n=0}^{\infty} n(n-1)a_n P(x) (x - x_o)^{n-2} + \sum_{n=0}^{\infty} n a_n Q(x) (x - x_o)^{n-1} + \sum_{n=0}^{\infty} a_n R(x) (x - 1)^n$$

Step 2. Unfortunately, depending on the nature of the polynomials, it may happen that none of three series in (25.3) is a power series in $(x - x_o)$. For example, if $P(x) = x^2$ and $x_o = 1$, then the first series is

$$(25.4) \quad \sum_{n=0}^{\infty} n(n-1)a_n x^2 (x-1)^2$$

which is not a power series (i.e., an expression of the form $\sum b_n (x-1)^n$ with each b_n a constant). To convert the series in (25.3) into to power series we must replace the polynomials $P(x)$, $Q(x)$, and $R(x)$ with their Taylor expansions about $x_o = 1$. If we set

$$(25.5) \quad \begin{aligned} p_n &= \frac{1}{n!} \frac{d^n P}{dx^n} (x_o) \\ q_n &= \frac{1}{n!} \frac{d^n Q}{dx^n} (x_o) \\ r_n &= \frac{1}{n!} \frac{d^n R}{dx^n} (x_o) \end{aligned}$$

we can write

$$(25.6) \quad \begin{aligned} P(x) &= \sum_{i=0}^{\infty} p_i (x - x_o)^i \quad , \\ Q(x) &= \sum_{i=0}^{\infty} q_i (x - x_o)^i \quad , \\ R(x) &= \sum_{i=0}^{\infty} r_i (x - x_o)^i \quad . \end{aligned}$$

Actually, since polynomial of degree D can have at most D non-vanishing derivatives, each of the Taylor expansions (25.6) will terminate after a finite number of terms:

$$(25.7) \quad \begin{aligned} P(x) &= \sum_{i=0}^{d_P} p_i (x - x_o)^i \quad , \\ Q(x) &= \sum_{i=0}^{d_Q} q_i (x - x_o)^i \quad , \\ R(x) &= \sum_{i=0}^{d_R} r_i (x - x_o)^i \quad . \end{aligned}$$

where d_P , d_Q , and d_R are the degrees of the polynomials $P(x)$, $Q(x)$, and $R(X)$. Inserting the expression (25.7) into (25.3) we get

$$\begin{aligned}
 0 &= \sum_{n=0}^{\infty} \sum_{i=0}^{d_P} n(n-1)a_n p_n (x-x_o)^{n+i-2} \\
 &\quad + \sum_{n=0}^{\infty} \sum_{i=0}^{d_P} n a_n q_n (x-x_o)^{n+i-1} \\
 &\quad + \sum_{n=0}^{\infty} \sum_{i=0}^{d_R} a_n r_n (x-x_o)^{n+i} \\
 (25.8) \qquad &= \sum_{i=0}^{d_P} \sum_{n=0}^{\infty} n(n-1)a_n p_n (x-x_o)^{n+i-2} \\
 &\quad + \sum_{i=0}^{d_Q} \sum_{n=0}^{\infty} n a_n q_n (x-x_o)^{n+i-1} \\
 &\quad + \sum_{i=0}^{d_R} \sum_{n=0}^{\infty} a_n r_n (x-x_o)^{n+i}
 \end{aligned}$$

or

$$\begin{aligned}
 0 &= \sum_{n=0}^{\infty} n(n-1)p_0 a_n (x-x_o)^{n-2} + \cdots + \sum_{n=0}^{\infty} n(n-1)p_{d_P} a_n (x-x_o)^{n+d_P-2} \\
 (25.9) \qquad &\quad + \sum_{n=0}^{\infty} n q_0 a_n (x-x_o)^{n-1} + \cdots + \sum_{n=0}^{\infty} n q_{d_Q} a_n (x-x_o)^{n+d_Q-1} \\
 &\quad + \sum_{n=0}^{\infty} r_0 a_n (x-x_o)^n + \cdots + \sum_{n=0}^{\infty} r_{d_R} a_n (x-x_o)^{n+d_R}
 \end{aligned}$$

Step 3. The next step is to collect all the terms consisting of like factors of $(x-x_o)^i$. To accomplish this we shift the summation index n in each series in (25.9) so that the k^{th} term in the new series has $(x-x_o)^k$ as a factor. One obtains

$$\begin{aligned}
 0 &= \sum_{k=-2}^{\infty} (k+2)(k+1)p_0 a_{k+2} (x-x_o)^k + \cdots \\
 &\quad \cdots + \sum_{n=-2+d_P}^{\infty} (k+2-d_P)(k+1-d_P)p_{d_P} a_{k+2-d_P} (x-x_o)^k \\
 &\quad + \sum_{k=0}^{\infty} (k+1)q_0 a_{k+1} (x-x_o)^k + \cdots + \sum_{k=-1+d_Q}^{\infty} (k+1-d_Q)q_{d_Q} a_{k+1-d_Q} (x-x_o)^k \\
 (25.10) \qquad &\quad + \sum_{k=0}^{\infty} r_0 a_k (x-x_o)^k + \cdots + \sum_{k=d_R}^{\infty} r_{d_R} a_{k-d_R} (x-x_o)^k
 \end{aligned}$$

Here one must be a bit careful. Notice that the various series appearing in the above equation **do not** have the same initial value of k . Before consolidating the various series in (25.10) in a single series we must make sure they all start off at the same value of k . I will discuss this point momentarily with an example. But certainly for k large enough all the series in (25.10) will contribute terms proportional to $(x-x_o)^k$. One can then read off from (25.10) the general recursion relation

$$\begin{aligned}
 0 &= (k+2)(k+1)p_0 a_{k+2} + \cdots + (k+2-d_P)(k+1-d_P)a_{k+2-d_P} \\
 (25.11) \qquad &\quad + (k+1)q_0 a_{k+1} + \cdots + (k+1-d_Q)q_{d_Q} a_{k+1-d_Q} \\
 &\quad + r_0 a_k + \cdots + r_{d_R} a_{k-d_R}
 \end{aligned}$$

which is valid for $k > \text{Max}\{-2+d_P, -1+d_Q, d_R\}$. Actually, we can use this relation for all k so long as we consistently define

$$(25.12) \qquad a_i = 0 \quad , \quad \text{if } i < 0.$$

Step 4. Use the recursion relation (25.11) to express all the coefficients a_n in terms of a_0 and a_1 (you may also need to use the relations $0 = a_{-1} = a_{-2} = a_{-3} \cdots$ coming from (25.12)).

EXAMPLE 25.1. Find a power series solution of

$$(25.13) \qquad x^2 y'' + (x+1)y = 0$$

about the point $x_o = 1$.

Plugging

$$(25.14) \qquad y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$$

into (25.13) yields

$$(25.15) \quad 0 = \sum_{n=0}^{\infty} n(n-1)a_n x^2 (x-1)^{n-2} + \sum_{n=0}^{\infty} a_n (x+1)(x-1)^n \quad .$$

Now the Taylor expansions of $f(x) = x^2$ and $g(x) = x+1$ about $x_0 = 1$ are

$$(25.16) \quad \begin{aligned} x^2 &= 1 + 2(x-1) + (x-1)^2 \\ x+1 &= 2 + (x-1) \quad . \end{aligned}$$

Plugging the right hand sides of (25.16) into (25.15) yields

$$(25.17) \quad \begin{aligned} 0 &= \sum_{n=0}^{\infty} n(n-1)a_n (1 + 2(x-1) + (x-1)^2) (x-1)^{n-2} \\ &\quad + \sum_{n=0}^{\infty} a_n (2 + (x-1)) (x-1)^n \\ &= \sum_{n=0}^{\infty} n(n-1)a_n (x-1)^{n-2} + \sum_{n=0}^{\infty} 2n(n-1)a_n (x-1)^{n-1} \\ &\quad + \sum_{n=0}^{\infty} n(n-1)a_n (x-1)^n + \sum_{n=0}^{\infty} 2a_n (x-1)^n + \sum_{n=0}^{\infty} a_n (x-1)^{n+1} \end{aligned}$$

We now shift the summation indices in each series so that in the k^{th} term, $(x-1)$ appears to the k^{th} power. One gets

$$(25.18) \quad 0 = 0 + 0 + \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}(x-1)^k + 0 + \sum_{k=0}^{\infty} 2(k+1)k a_{k+1}(x-1)^k \\ + \sum_{k=0}^{\infty} k(k-1)a_k (x-1)^k + \sum_{k=0}^{\infty} 2a_k (x-1)^k + \sum_{k=1}^{\infty} a_{k-1}(x-1)^k$$

Unfortunately, the last series begins with $k = 1$, instead of $k = 0$. This, however, is easy to remedy; we simply $a_{-1} = 0$, so that

$$(25.19) \quad \sum_{k=0}^{\infty} a_{k-1}(x-1)^k = 0(x-k)^{-1} + \sum_{k=1}^{\infty} a_{k-1}(x-1)^k = \sum_{k=1}^{\infty} a_{k-1}(x-1)^k \quad .$$

Thus, having arranged things so that all series start off at the same point $k = 0$ and we now consolidate the right hand side of (25.18) into a single series:

$$(25.20) \quad \begin{aligned} 0 &= \sum_{k=0}^{\infty} ((k+2)(k+1)a_{k+2} + 2(k+1)k a_{k+1} + k(k-1)a_k + 2a_k + a_{k-1}) (x-1)^k \\ &= \sum_{k=0}^{\infty} ((k+2)(k+1)a_{k+2} + 2k(k+1)a_{k+1} + (k^2 - k + 2)a_k + a_{k-1}) (x-1)^k \end{aligned}$$

The demand that the total coefficient of $(x-1)^k$ vanish then implies

$$(25.21) \quad a_{k+2} = \frac{-2k(k+1)a_{k+1} - (k^2 - k + 2)a_k - a_{k-1}}{(k+2)(k+1)} \quad .$$

Thus, given that $a_{-1} = 0$, we have

$$(25.22) \quad \begin{aligned} a_2 &= \frac{0-2a_0-0}{(25.2)(25.1)} = -a_0 \\ a_3 &= \frac{(-2)(25.3)a_2 - (25.2)a_1 - a_0}{(25.3)(25.2)} = \frac{-7a_0 - 2a_1}{6} \\ a_4 &= \frac{(-4)(25.3)a_3 - 4a_2 - a_1}{(25.4)(25.3)} = \frac{(14a_0 - 4a_1 + 4a_0 - a_1)}{12} = \frac{18a_0 - 5a_1}{12} \end{aligned}$$

Thus, to the order of $(x-1)^4$ the general solution of (25.13) is

$$(25.23) \quad \begin{aligned} y(x) &= a_0 + a_1(x-1) - a_0(x-1)^2 - \frac{7a_0 - 2a_1}{6}(x-1)^3 \\ &\quad + \frac{18a_0 - 5a_1}{12}(x-1)^4 + \dots \\ &= a_0 \left(1 - (x-1)^2 - \frac{7}{6}(x-1)^3 + \frac{3}{2}(x-1)^4 + \dots \right) \\ &\quad + a_1 \left((x-1) + \frac{1}{3}(x-1)^3 - \frac{5}{12}(x-1)^4 + \dots \right) \end{aligned}$$