Sample Second Exam
November 5, 1997

1.(15 pts) Use a change of variable to solve the following first order equation. (Hint: note that it is homogeneous of degree zero).

$$y' = \frac{xy + y^2}{x^2}$$

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$$\frac{dy}{dx} = \left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2$$

Under the substitution $u = y/x$, or equivalently $y = xu$, we have

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

Therefore, our original differential equation is equivalent to

$$u + x\frac{du}{dx} = \frac{dy}{dx} = \left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 = u + u^2$$

Cancelling the terms u that appear on the extreme sides of this equation yields

$$x\frac{du}{dx} = u^2$$

or

$$\frac{du}{u^2} = \frac{dx}{x}$$

Integrating both sides yields

$$-\frac{1}{u} = \ln|x| + C$$

Now we recall $u = y/x$ to get

$$-\frac{x}{y} = \ln|x| + C$$

or

$$y(x) = -\frac{x}{\ln|x| + C}$$

∎

2. Given that $y_1(x) = x$ and $y_2(x) = x^3$ are solutions to $x^2y'' - 3xy' + 3y = 0$

(a) (5 pts) Show that the functions $y_1(x)$ and $y_2(x)$ are linearly independent.

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$$W[y_1, y_2](x) = (x)(3x^2) - (1)(x^3) = 2x^3 \neq 0$$

so $y_1(x)$ and $y_2(x)$ must be linearly independent. ∎

(b) (5 pts) Write down the general solution.

- $$y(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 x + c_2 x^3$$

  ▮

(c) (5 pts) Find the solution satisfying the initial conditions $y(1) = 1$, $y'(1) = 1$.

- $$\begin{aligned} 1 &= y(1) = c_1(1) + c_2(1^2) = c_1 + c_2 \\ 1 &= y'(1) = c_1 + c_2\left(3x^2\right)\big|_{x=1} = c_1 + 3c_2 \end{aligned}$$

  Subtracting the second equation from the first yields

  $$0 = 0 - 2c_2 \quad \Rightarrow \quad c_2 = 0$$

  But then the first equation implies $c_1 = 1$. Thus,

  $$\begin{aligned} c_1 &= 1 \\ c_2 &= 0 \end{aligned}$$

  and the solution satisfying the given intitial conditions is

  $$y(x) = x$$

  ▮

3. (10 pts)Given that $y_1(x) = x^2$ is one solution of $x^2 y'' - 4xy' + 6y = 0$, use Reduction of Order to determine the general solution.

- Putting the differential equation in standard form we see that the term $p(x)$ in the Reduction of Order formula is $p(x) = -\frac{4}{x}$. Thus, a second linearly independent solution is

  $$\begin{aligned} y_2(x) &= y_1(x) \int^x \frac{1}{(y_1(s))^2} \exp\left[-\int^s p(t)dt\right] ds \\ &= x^2 \int^x \frac{1}{s^4} \exp\left[+4\int^s \frac{dt}{t}\right] ds \\ &= x^2 \int^x s^{-4} \exp\left(4 \ln|s|\right) ds \\ &= x^2 \int^x s^{-4} s^4 ds \\ &= x^2 \int^x ds \\ &= x^2(x) \\ &= x^3 \end{aligned}$$

  Now that we have two linearly independent solutions, we can write down the general solution:

  $$y(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 x^2 + c_2 x^3$$

  ▮

4. Determine the general solution of the following differential equations.

(a) (5 pts) $y'' - 3y' - 3y = 0$

- The characteristic equation for this homogeneous linear equation with constant coefficients is

$$\lambda^2 - 3\lambda - 3 = 0.$$

The roots of this equation are determined by the Quadratic Formula

$$\lambda = \frac{3 \pm \sqrt{9-(4)(-3)}}{2} = \frac{3 \pm \sqrt{21}}{2}$$

So

$$\begin{aligned} y_1(x) &= e^{\frac{3+\sqrt{21}}{2}x} \\ y_2(x) &= e^{\frac{3-\sqrt{21}}{2}x} \end{aligned}$$

and the general solution is

$$y(x) = c_1 e^{\frac{3+\sqrt{21}}{2}x} + c_2 e^{\frac{3-\sqrt{21}}{2}x}$$

∎

(b) (5 pts) $y'' + 10y' + 25y = 0$

- The characteristic equation for this homogeneous linear equation with constant coefficients is

$$0 = \lambda^2 + 10\lambda + 25 = (\lambda + 5)^2$$

and so we have a single root $\lambda = -5$. The two linearly independent solutions are thus

$$\begin{aligned} y_1(x) &= e^{-5x} \\ y_2(x) &= xe^{-5x} \end{aligned}$$

and the general solution is

$$y(x) = c_1 e^{-5x} + c_2 x e^{-5s}$$

∎

(c) (5 pts) $y'' - 4y' + 13y = 0$

- The characteristic equation for this homogeneous linear equation with constant coefficients is

$$\lambda^2 - 4\lambda + 13 = 0.$$

Applying the Quadratic Formula we obtain

$$\lambda = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm -6}{2} = 2 \pm 3i$$

We thus have a pair of complex roots. The corresponding linearly independent (real-valued) solutions are

$$\begin{aligned} y_1(x) &= e^{2x}\cos(3x) \\ y_2(x) &= e^{2x}\sin(3x) \end{aligned}$$

and so the general solution is

$$y(x) = c_1 e^{2x}\cos(3x) + c_2 e^{2x}\sin(3x)$$

∎

5. (10 pts) Explain in words how one could use Reduction of Order and the Method of Variation of Parameters to construct the general solution of $x^2y'' - 2y = 3x^2 - 1$, given that $y_1(x) = x^2$ is a solution of $x^2y'' - 2y = 0$. (It is not necessary to carry out any of the calculations.)

- First we'd use the Reduction of Order formula

$$y_2(x) = y_1(x) \int^x \frac{1}{(y_1(s))^2} \exp\left[-\int^s p(t)dt\right] ds$$

to compute a second linearly independent solution $y_2(x)$ of $x^2y'' - 2y = 0$. Next we'd use the Variation of Parameters formula

$$y_p(x) = -y_1(x) \int^x \frac{y_2(s)g(s)}{W[y_1, y_2](s)} ds + y_2(x) \int^x \frac{y_1(s)g(s)}{W[y_1, y_2](s)} ds$$

with $g(s) = (3x^2 - 1)/x^2$ to construct a particular solution of $x^2y'' - 2y = 3x^2 - 1$. We'd then have all the ingredients necessary to write down the general solution:

$$y(x) = y_p(x) + c_1y_1(x) + c_2y_2(x)$$

∎

6. Given that $y_1(x) = e^x$ and $y_2(x) = e^{2x}$ are solutions of $y'' - 3y' + 2y = 0$.

(a) (10 pts) Use the Method of Variation of Parameters to find a particular solution of

$$y'' - 3y' + 2y = e^x \quad .$$

- Before applying the Variation of Parameters formula we note that

$$W[y_1, y_2](x) = (e^x)\left(2e^{2x}\right) - (e^x)\left(e^{2x}\right) = e^{3x}$$

and

$$g(x) = e^x$$

We can now compute a particular solution to the non-homogeneous equation

$$\begin{aligned} y_p(x) &= -y_1(x) \int^x \frac{y_2(s)g(s)}{W[y_1, y_2](s)} ds + y_2(x) \int^x \frac{y_1(s)g(s)}{W[y_1, y_2](s)} ds \\ &= -e^x \int^x \frac{e^{2s}(e^s)}{e^{3s}} ds + e^{2x} \int^x \frac{e^s(e^s)}{e^{3s}} ds \\ &= -e^x \int^x ds + e^{2x} \int e^{-s} ds \\ &= -e^x(x) + e^{2x}\left(-e^{-x}\right) \\ &= -xe^x - e^x \\ &\approx -xe^x \end{aligned}$$

(The second term can be ignored since it is a solution to the corresponding homogeneous equation.) ∎

(b) (10 pts) Find the solution satisfying $y(0) = 0$, $y'(0) = 2$.

- The general solution to the non-homogeneous problem in part (a) is

$$y(x) = y_p(x) + c_1y_1(x) + c_2y_2(x) = -xe^x + c_1e^x + c_2e^{2x}$$

Applying the initial conditions yields

$$\begin{aligned} 0 &= y(0) = -(0)e^0 + c_1e^0 + c_2e^0 = c_1 + c_2 \\ \\ 2 &= y'(0) = -e^x - xe^x + c_1e^x + 2c_2e^{2x}\big|_{x=0} = -1 + c_1 + 2c_2 \end{aligned}$$

We thus have

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 + 2c_2 &= 3 \end{aligned}$$

Using the first equation we can substitute $c_1 = -c_2$ into the second to obtain

$$c_2 = 3$$

from which we can conclude also that $c_1 = 3$. Hence the solution to the initial value problem is

$$y(x) = -xe^x - 3e^x + 3e^{2x}$$

∎

7. Find the general solution of the following Euler-type differential equations.

(a) (5 pts) $x^2y'' + 4xy' + y = 0$

- Substituting $y(x) = x^r$ into the differential equation yields

$$(r(r-1) + 4r + 1)\, x^r = 0$$

so we must have

$$0 = r(r-1) + 4r + 1 = r^2 + 3r + 1$$

Applying the Quadratic Formula yields

$$r = \frac{-3 \pm \sqrt{9 - (4)(1)}}{2} = \frac{-3 \pm \sqrt{5}}{2}$$

Hence we have the following two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{\frac{-3+\sqrt{5}}{2}} \\ y_2(x) &= x^{\frac{-3-\sqrt{5}}{2}} \end{aligned}$$

and the general solution is

$$y(x) = c_1 x^{\frac{-3+\sqrt{5}}{2}} + c_2 x^{\frac{-3-\sqrt{5}}{2}}$$

∎

(b) (5 pts) $x^2y'' - 5xy' + 9y = 0$

- Substituting $y(x) = x^r$ into the differential equation yields

$$(r(r-1) - 5r + 9)\, x^r = 0$$

so we must have

$$0 = r(r-1) - 5r + 9 = r^2 - 6r + 9 = (r-3)^2$$

We thus have a single root $r = 3$ and thus two linearly independent solutions will be

$$\begin{aligned} y_1(x) &= x^{-3} \\ y_2(x) &= x^{-3} \ln|x| \end{aligned}$$

The general solution is thus

$$y(x) = c_1 x^{-3} + c_2 x^{-3} \ln|x|$$

∎

(c) (5 pts) $x^2y'' - 5xy' + 13y = 0$

- Substituting $y(x) = x^r$ into the differential equation yields

$$(r(r-1) - 5r + 13)\, x^r = 0$$

so we must have

$$0 = r(r-1) - 5r + 13 = r^2 - 6r + 13$$

Applying the Quadratic Formula we see that

$$r = \frac{6 \pm \sqrt{36 - (4)(13)}}{2} = \frac{6 \pm \sqrt{36 - 52}}{2} = \frac{-6 \pm \sqrt{-16}}{2} = \frac{6 \pm 4i}{2} = 3 \pm 2i$$

We thus have the following (real-valued) linearly independent solutions

$$\begin{aligned} y_1(x) &= x^3 \cos(2 \ln |x|) \\ y_2(x) &= x^3 \sin(2 \ln |x|) \end{aligned}$$

and so the general solution is

$$y(x) = c_1 x^3 \cos(2 \ln |x|) + c_2 x^3 \sin(2 \ln |x|)$$

∎