LECTURE 18

Euler Equations

We are now going to consider how to construct solutions of a slightly broader class of differential equations; those of the form

$$x^2 y'' + \alpha x y' + \beta y = 0 \quad , \tag{18.1}$$

where α and β are constants. Note that the point $x = 0$ is a regular singular point. A differential equation of this form is called an **Euler equation**.

To solve such equations, we make the following *ansatz*:

$$y(x) = x^r \quad . \tag{18.2}$$

Then

$$\begin{aligned} y' &= r x^{r-1} \\ y'' &= r(r-1)x^{r-2} \end{aligned} \tag{18.3}$$

and so plugging (18.2) into (18.1) yields

$$\begin{aligned} 0 &= x^2\left(r(r-1)x^{r-2}\right) + \alpha x\left(r x^{r-1}\right) + \beta x^r \\ &= \left(r(r-1) + \alpha r + \beta\right) x^r \\ &= \left(r^2 + (\alpha - 1)r + \beta\right) x^r \quad . \end{aligned} \tag{18.4}$$

We can thus ensure that (18.2) is a solution of (18.1) by demanding

$$r^2 + (\alpha - 1)r + \beta = 0 \tag{18.5}$$

or

$$r = \frac{1 - \alpha \pm \sqrt{(1-\alpha)^2 - 4\beta}}{2} \quad . \tag{18.6}$$

Like that the case of second order differential equations with constant coefficients, we have three different kinds of solutions, depending on the nature of the quantity inside the square root.

Case (i): $(1-\alpha)^2 - 4\beta > 0$.

In this case the number inside the radical is positive, so we find a (real) square root. We end up with two distinct roots

$$\begin{aligned} r_+ &= \frac{1-\alpha+\sqrt{(1-\alpha)^2-4\beta}}{2} \\ r_- &= \frac{1-\alpha-\sqrt{(1-\alpha)^2-4\beta}}{2} \end{aligned} \tag{18.7}$$

and, accordingly, two linearly independent solutions

$$y_1(x) = x^{r_+} \quad , \quad y_2(x) = x^{r_-} \quad . \tag{18.8}$$

The general solution is thus

$$y(x) = c_1 x^{r_+} + c_2 x^{r_-} \quad . \tag{18.9}$$

Case (ii): $(1-\alpha)^2 - 4\beta = 0$.

In this case, we only have one distinct root

$$r = \frac{1-\alpha}{2} \tag{18.10}$$

and so obtain only one distinct solution

$$y_1(x) = x^r = x^{\frac{1-\alpha}{2}} \quad . \tag{18.11}$$

A second linearly independent solution however may be found using reduction of order:

$$\begin{aligned} y_2(x) &= y_1(x) \int^x \frac{1}{(y_1(t))^2} \exp\left(-\int^t p(s)ds\right) dt \\ &= x^{\frac{1-\alpha}{2}} \int^x t^{-1+\alpha} \exp\left(-\int^t \frac{\alpha}{s} ds\right) dt \\ &= x^{\frac{1-\alpha}{2}} \int^x t^{-1+\alpha} \exp\left(\ln |t^{-\alpha}|\right) \\ &= x^{\frac{1-\alpha}{2}} \int^x t^{-1} dt \\ &= x^{\frac{1-\alpha}{2}} \ln |x| \quad . \end{aligned} \tag{18.12}$$

So in this case the general solution is

$$y(x) = c_1 x^{\frac{1-\alpha}{2}} + c_2 x^{\frac{1-\alpha}{2}} \ln |x| \quad . \tag{18.13}$$

Case (iii): $(1-\alpha)^2 - 4\beta < 0$.

In this case the quantity inside the radical is negative so the roots of (18.1) are complex numbers. We set

$$\lambda = \frac{1-\alpha}{2} \quad , \quad \mu = \frac{\sqrt{4\beta - (1-\alpha)^2}}{2} \tag{18.14}$$

so that we can write the roots of (18.1) as

$$r_{\pm} = \lambda \pm i\mu \tag{18.15}$$

and write the general solution as

$$y(x) = c_1 x^{\lambda + i\mu} + c_2 x^{\lambda - i\mu} \quad . \tag{18.16}$$

However, we still have to make sense of x raised to a complex power. This is done as follows:

$$\begin{aligned} x^{\lambda+i\mu} &= \left(\exp\left(\ln |x|\right)\right)^{\lambda+i\mu} \\ &= \left(\exp\left(\ln |x|\right)\right)^{\lambda} \left(\exp\left(\ln |x|\right)\right)^{i\mu} \\ &= x^{\lambda} \left(\exp\left(i\mu \ln |x|\right)\right) \\ &= x^{\lambda} \left(\cos(\mu \ln |x|) + i \sin(\mu \ln |x|)\right) \end{aligned} \tag{18.17}$$

The real and imaginary parts of this solution will also be solutions, and, in fact, they will constitute a fundamental set of real-valued solutions to (**??**). Thus, in this case the general solution will be

$$y(x) = c_1 x^{\lambda} \cos\left(\mu \ln |x|\right) + c_2 x^{\lambda} \sin\left(\mu \ln |x|\right) \quad . \tag{18.18}$$

The table below reviews the construction of solutions to Euler type equations and at the same time shows its similarity with the construction of solutions of 2^{nd} order linear differential equations with constant coefficients.

Comparison between Euler-Type Equations and Equations with Constant Coefficients

	Euler-Type	**Constant Coeffficients**
Equation:	$y'' + \frac{\alpha}{x} y' + \frac{\beta}{x^2} y = 0$	$ay'' + by' + cy = 0$
Ansatz:	$y(x) = x^r$	$y(x) = e^{\lambda x}$
Condition on r, λ:	$r^2 + (\alpha - 1) r + \beta = 0$ $r_\pm = \frac{1-\alpha \pm \sqrt{(\alpha-1)^2 - 4\beta}}{2}$	$a\lambda^2 + b\lambda + c = 0$ $\lambda_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Case (i)	$(\alpha - 1)^2 - 4\beta > 0$ $\Rightarrow y(x) = c_1 x^{r_+} + c_2 x^{r_-}$	$b^2 - 4ac > 0$ $\Rightarrow y(x) = c_1 e^{\lambda_+ x} + c_2 e^{\lambda_- x}$
Case (ii)	$(\alpha - 1)^2 - 4\beta = 0$ $r = \frac{1-\alpha}{2}$ $\Rightarrow y(x) = c_1 x^r + c_2 x^r \ln\|x\|$	$b^2 - 4ac = 0$ $\lambda = \frac{-b}{2a}$ $\Rightarrow y(x) = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$
Case (iii)	$(\alpha - 1)^2 - 4\beta < 0$ $r = \lambda \pm i\mu$ $\Rightarrow y(x) = c_1 x^\lambda \cos(\mu \ln\|x\|)$ $+ c_2 x^\lambda \sin(\mu \ln\|x\|)$	$b^2 - 4ac < 0$ $\lambda = \alpha \pm i\beta$ $\Rightarrow y(x) = c_1 e^{\alpha x} \cos(\beta x)$ $+ c_2 e^{\alpha x} \sin(\beta x)$

EXAMPLE 18.1. $x^2 y'' - 2xy' + 2y = 0$

Substituting $y(x) = x^r$ into this differential equation yields

$$r(r-1)x^r - 2\left(rx^r\right) + 2x^r = 0$$

or

$$\left(r^2 - r - 2r + 2\right) x^r = 0$$

so we must have

$$0 = r^2 - r - 2r + 2 = r^2 - 3r + 2 = (r-2)(r-1)$$

Thus, we have $r = 2, 1$. The general solution is thus

$$y(x) = c_1 x^2 + c_2 x^1$$

EXAMPLE 18.2. $x^2 y'' + 7xy' + 9y = 0$

Substituting $y(x) = x^r$ into this differential equation yields

$$r(r-1)x^r + 7\left(rx^r\right) + 9x^r = 0$$

or

$$\left(r^2 - r + 7r + 9\right) x^r = 0$$

so we must have

$$0 = r^2 - r + 7r + 9 = r^2 + 6r + 9 = (r+3)^2$$

Thus, we have only a single root of the indicial equation $r = -3$. The general solution is thus

$$y(x) = c_1 x^{-3} + c_2 \ln|x| x^{-3}$$

EXAMPLE 18.3. $x^2y'' + xy' + 4y = 0$

Substituting $y(x) = x^r$ into this differential equation yields

$$r(r-1)x^r + (rx^r) + 4x^r = 0$$

or

$$\left(r^2 - r + r + 4\right)x^r = 0$$

so we must have

$$0 = r^2 - r + r + 4 = r^2 + 4 = (r+2i)(r-2i)$$

Thus, we have a pair of complex roots $r = 0 + 2i, 0 - 2i$. The general solution is thus

$$\begin{aligned} y(x) &= c_1x^0\cos(2\ln|x|) + c_2x^0\sin(2\ln|x|) \\ &= c_1\cos(2\ln|x|) + c_2\sin(2\ln|x|) \end{aligned}$$