LECTURE 17

Homogeneous Equations with Constant Coefficients, Cont'd

Recall that the general solution of a 2^{nd} order linear homogeneous differential equation

$$L[y] = y'' + p(x)y' + q(x)y = 0 \tag{17.1}$$

is always a linear combination

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \tag{17.2}$$

of two linearly independent solutions y_1 and y_2, and we've seen that if we're given one solution $y_1(x)$ we can compute a second linearly independent solution using the method of reduction of order. We will now turn to the problem of actually finding a single solution $y_1(x)$ of (17.1).

We let us now return to the special case of a homogeneous second order linear differential equation with constant coefficients; i.e., differential equations of the form

$$y'' + py' + qy = 0 \tag{17.3}$$

where p and q are constant.

We saw in Lecture 11, that one can construct solutions of the differential equation (17.3) by looking for solutions of the form

$$y(x) = e^{\lambda x} \quad . \tag{17.4}$$

Let us recall that construction. Plugging (17.4) into (17.3) yields

$$0 = \lambda^2 e^{\lambda x} + p\lambda e^{\lambda x} + qe^{\lambda x} = \left(\lambda^2 + p\lambda + q\right) e^{\lambda x} \quad . \tag{17.5}$$

Since the exponential function $e^{\lambda x}$ never vanishes we must have

$$\lambda^2 + p\lambda + q = 0 \quad . \tag{17.6}$$

Equation (17.6) is called the **characteristic equation** for (17.3) since for any λ satisfying (17.6) we will have a solution $y(x) = e^{\lambda x}$ of (17.3).

Now because (17.6) is a quadratic equation we can employ the Quadratic Formula to find all of its roots:

$$\lambda^2 + p\lambda + q = 0 \qquad \Rightarrow \qquad \lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2} \quad . \tag{17.7}$$

Note that a root λ of (17.6) need not be a real number. Indeed, if $p^2 - 4q < 0$, then in order to compute λ via the Quadratic Formula we have to take the square root of a negative number and that forces us into the realm of complex numbers. We shall postpone temporarily the case when a root λ is complex and first discuss the case when the roots of (17.6) are all real. This requires $p^2 - 4q \geq 0$.

Case (i): $p^2 - 4q > 0$

Because $p^2 - 4q$ is positive, $\sqrt{p^2-4q}$ is a positive real number and

$$\begin{aligned} \lambda_+ &= \tfrac{-p+\sqrt{p^2-4q}}{2} \\ \lambda_- &= \tfrac{-p-\sqrt{p^2-4q}}{2} \end{aligned} \tag{17.8}$$

are distinct real roots of (17.6). Thus,

$$\begin{aligned} y_1 &= e^{\lambda_+ x} \\ y_2 &= e^{\lambda_- x} \end{aligned} \tag{17.9}$$

will both be solutions of (17.3). Noting that

$$\begin{aligned} W(y_1,y_2) &= y_1 y_2' - y_1' y_2 \\ &= \lambda_- e^{\lambda_+ x} e^{\lambda_- x} - \lambda_+ e^{\lambda_+ x} e^{\lambda_- x} \\ &= (\lambda_- - \lambda_+)\, e^{(\lambda_+ + \lambda_-)x} \\ &= \tfrac{\sqrt{p^2-4q}}{a} e^{-\frac{b}{a}x} \end{aligned} \tag{17.10}$$

is non-zero, we conclude that if $p^2 - 4q \neq 0$, then the roots (17.8) furnish two linearly independent solutions of (17.3) and so the general solution is given by

$$y(x) = c_1 e^{\lambda_+ x} + c_2 e^{\lambda_- x} \quad . \tag{17.11}$$

Case (ii): $p^2 - 4q = 0$

If $p^2 - 4q = 0$, however, this construction only gives us one distinct solution; because in this case $\lambda_+ = \lambda_-$. To find a second fundamental solution we must use the method of Reduction of Order.

So suppose $y_1(x) = e^{-\frac{p}{2}x}$ is the solution corresponding to the root

$$\lambda = \frac{-p \pm \sqrt{p^2-4q}}{2} = \frac{-p \pm 0}{2} = \frac{-p}{2}$$

of

$$\lambda^2 + p\lambda - q = 0 \quad , \quad p^2 - 4q = 0.$$

Then the Reduction of Order formula gives us a second linearly independent solution

$$y_2(x) = y_1(x) \int^x \frac{1}{(y_1(s))^2} \exp\left[\int^s -p(t)dt\right] ds$$

gives us a second linearly independent solution. Plugging in $y_1(x) = e^{-\frac{p}{2}x}$ and $p(t) = p$, yields

$$\begin{aligned} y_2(x) &= e^{-\frac{p}{2}x} \int^x \frac{1}{\left(e^{-\frac{p}{2}s}\right)^2} \exp\left[\int^s -pdt\right] ds \\ &= e^{-\frac{p}{2}x} \int^x \frac{1}{e^{-ps}} \exp\left[-ps\right] ds \\ &= e^{-\frac{p}{2}x} \int^x e^{ps} e^{-ps} ds \\ &= e^{-\frac{p}{2}x} \int^x ds \\ &= x e^{-\frac{p}{2}x} \\ &= x y_1(x) \end{aligned}$$

In summary, for the case when $p^2 - 4q = 0$, we only have one root of the characterisitic equation, and so we get only one distinct solution $y_1(x)$ of the original differential equation by solving the characteristic equation for λ. To get a second linearly solution we must use the Reduction of Order formula; however, the result will always be the same: **the second linearly independent solution will always be x times the solution** $y_1(x) = e^{-\frac{p}{2}x}$. Thus, the general solution in this case will be

$$y(x) = c_1 e^{-\frac{p}{2}x} + c_2 x e^{-\frac{p}{2}x} \quad , \quad \text{if } p^2 - 4q = 0.$$

We now turn to the third and last possibility.

Case (iii): $p^2 - 4q < 0$

In this case

$$\sqrt{p^2 - 4q} \tag{17.12}$$

will be undefined unless we introduce complex numbers. But when we set

$$\sqrt{-1} = i \tag{17.13}$$

we have

$$\sqrt{p^2 - 4q} = \sqrt{(-1)(4q - p^2)} = \sqrt{-1}\sqrt{4q - p^2} = i\sqrt{4q - p^2} \quad . \tag{17.14}$$

The square root on the right hand side is well-defined since $4q - p^2$ is a positive number. Thus,

$$\lambda_{\pm} = \frac{-p \pm i\sqrt{4q - p^2}}{2} = \alpha \pm i\beta \tag{17.15}$$

where

$$\alpha = -\frac{b}{2} \quad , \quad \beta = \frac{\sqrt{4q - p^2}}{2} \quad , \tag{17.16}$$

will be a complex solution of (17.6) and

$$y(x) = c_1 e^{\alpha x + i\beta x} + c_2 e^{\alpha x - i\beta x} \tag{17.17}$$

would be a solution of (17.3) if we could make sense out the notion of an exponential function with a complex argument.

Thus, we must address the problem of ascribing some meaning to

$$e^{\alpha x + i\beta x} \tag{17.18}$$

as a function of x. To ascribe some sense to this expression we considered the Taylor series expansion of e^x

$$\begin{aligned} e^x &= 1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \cdots \\ &= \textstyle\sum_{i=0}^{\infty} \tfrac{1}{i!}x^i \end{aligned} \tag{17.19}$$

Now although we do not yet understand what $e^{\alpha x + i\beta x}$ means, we can nevertheless substitute $\alpha x + i\beta$ for x on the right hand side of (17.19), and get a well defined series with values in the complex numbers. One can show that this series converges for all α, β and x. We thus take

$$e^{\alpha x + i\beta} = \lim_{n \to \infty} \sum_{i=0}^{n} \frac{1}{i!}(\alpha x + i\beta x)^i \tag{17.20}$$

which agrees with (17.19) when $\beta = 0$.

One can also show that

$$e^{\alpha x + i\beta x} = e^{\alpha x} e^{i\beta x}. \tag{17.21}$$

Thus, when $p^2 - 4q = 0$, we have two complex valued solutions to (17.3)

$$y_1(x) = e^{\alpha x} e^{i\beta x} \quad \text{and} \quad y_2(x) = e^{\alpha x} e^{-i\beta x} \quad , \tag{17.22}$$

where

$$\alpha = \frac{-p}{2} \quad , \quad \beta = \frac{\sqrt{4q-p^2}}{2} \quad . \tag{17.23}$$

A general solution of (17.3) would then be

$$y(x) = c_1 e^{\alpha x} e^{i\beta x} + c_2 e^{\alpha x} e^{i\beta x}. \tag{17.24}$$

However, this is rarely the form in which one wants a solution of (17.3). One would prefer solutions that are **real-valued functions of** x rather that complex-valued functions of x. But these can be had as well, since if $z = x + iy$ is a complex number, then

$$\begin{aligned} Re(z) &= \tfrac{1}{2}(z+\bar{z}) = x \\ Im(z) &= \tfrac{1}{2i}(z-\bar{z}) = y \end{aligned} \tag{17.25}$$

are both real numbers. Applying the Superposition Principle, it is easy to see that if

$$y(x) = e^{\alpha x} e^{i\beta x} \tag{17.26}$$

and

$$\bar{y}(x) = e^{\alpha x} e^{-i\beta x} \tag{17.27}$$

are two complex-valued solutions of (17.3), then

$$y_r(x) = \frac{1}{2}(y(x) + \bar{y}(x)) = e^{\alpha x}\left(\frac{e^{i\beta x} + e^{-i\beta x}}{2}\right) \tag{17.28}$$

and

$$y_i(x) = \frac{1}{2i}(y(x) - \bar{y}(x)) = e^{\alpha x}\left(\frac{e^{i\beta x} - e^{-i\beta x}}{2i}\right) \tag{17.29}$$

are both real-valued solutions of (17.3).

Let us now compute the series expansion of

$$\frac{e^{ix} + e^{-ix}}{2} \tag{17.30}$$

and

$$\frac{e^{ix} - e^{-ix}}{2i} \quad . \tag{17.31}$$

$$\begin{aligned} \tfrac{1}{2}\left(e^{ix} + e^{-ix}\right) &= \tfrac{1}{2}\left(1 + (ix) + \tfrac{1}{2!}(ix)^2 + \tfrac{1}{3!}(ix)^3 + \cdots\right) \\ &\quad + \tfrac{1}{2}\left(1 + (-ix) + \tfrac{1}{2!}(-ix)^2 + \tfrac{1}{3!}(-ix)^3 + \cdots\right) \\ &= \left(1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 + \cdots\right) \end{aligned} \tag{17.32}$$

The expression on the right hand side is readily identified as the Taylor series expansion of $\cos(x)$. We thus conclude

$$\boxed{\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \quad .} \tag{17.33}$$

Similarly, one can show that

$$\boxed{\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \quad .} \tag{17.34}$$

On the other hand, if one adds (17.33) to i times (17.34) one gets

$$\cos(x) + i\sin(x) = \frac{e^{ix} + e^{-ix}}{2} + i\frac{e^{ix} - e^{-ix}}{2i} = \frac{e^{ix} + e^{-ix} + e^{ix} - e^{-ix}}{2} = e^{ix} \tag{17.35}$$

or

$$\boxed{e^{ix} = \cos(x) + i\sin(x)} \tag{17.36}$$

Thus, the real part of e^{ix} is $\cos(x)$, while the pure imaginary part of e^{ix} is $\sin(x)$.

We now have a means of interpreting the function

$$e^{\alpha x + i\beta x} \tag{17.37}$$

in terms of elementary functions (rather than as a power series); namely,

$$e^{\alpha x + i\beta x} = e^{\alpha x}e^{i\beta x} = e^{\alpha x}\left(\cos(\beta x) + i\sin(\beta x)\right). \tag{17.38}$$

Thus,

$$\boxed{\begin{array}{rcl} Re\left[e^{\alpha x+i\beta x}\right] & = & e^{\alpha x}\cos(\beta x) \quad , \\ Im\left[e^{\alpha x+i\beta x}\right] & = & e^{\alpha x}\sin(\beta x) \quad . \end{array}} \tag{17.39}$$

I now want to show how (17.33) and (17.34) allow us to write down the general solution of a differential equation of the form

$$y'' + py' + qy = 0 \quad , \quad p^2 - 4q < 0 \tag{17.40}$$

as a linear combination of real-valued functions.

Now when $p^2 - 4q < 0$, then

$$\lambda_{\pm} = \frac{-p \pm i\sqrt{4q - p^2}}{2} = \alpha \pm i\beta \tag{17.41}$$

are the (complex) roots of the characteristic equation

$$\lambda^2 + p\lambda + q = 0 \tag{17.42}$$

corresponding to (17.40) and

$$y_{\pm}(x) = e^{\alpha x \pm i\beta} \tag{17.43}$$

are two (complex-valued) solutions of (17.40). But since (17.40) is linear, since y_+ and y_- are solutions so are

$$\begin{array}{rcl} y_1(x) & = & \frac{1}{2}\left(y_+(x) + y_-(x)\right) \\ & = & \frac{1}{2}\left(e^{\alpha x + i\beta x} + e^{\alpha x - i\beta x}\right) \\ & = & e^{\alpha x}\left(\frac{e^{i\beta x} + e^{-i\beta x}}{2}\right) \\ & = & e^{\alpha x}\cos(\beta x) \end{array} \tag{17.44}$$

and

$$\begin{array}{rcl} y_2(x) & = & \frac{1}{2i}\left(y_+(x) - y_-(x)\right) \\ & = & \frac{1}{2i}\left(e^{\alpha x + i\beta x} - e^{\alpha x - i\beta x}\right) \\ & = & e^{\alpha x}\left(\frac{e^{i\beta x} - e^{-i\beta x}}{2i}\right) \\ & = & e^{\alpha x}\sin(\beta x) \quad . \end{array} \tag{17.45}$$

Note that y_1 and y_2 are both **real-valued functions**.

We conclude that if the characteristic equation corresponding to

$$y'' + py' + qy = 0 \tag{17.46}$$

has two complex roots

$$\lambda = \alpha \pm i\beta \tag{17.47}$$

then the general solution is

$$\boxed{y(x) = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x) \quad .} \tag{17.48}$$

EXAMPLE 17.1. The differential equation

$$y'' - 2y' - 3y \tag{17.49}$$

has as its characteristic equation

$$\lambda^2 - 2\lambda - 3 = 0 \quad . \tag{17.50}$$

The roots of the characteristic equation are given by

$$\begin{aligned} \lambda &= \tfrac{2\pm\sqrt{4+12}}{2} \\ &= 3, -1 \quad . \end{aligned} \tag{17.51}$$

These are distinct real roots, so the general solution is

$$y(x) = c_1 e^{3x} + c_2 e^{-x} \quad . \tag{17.52}$$

EXAMPLE 17.2. The differential equation

$$y'' + 4y' + 4y = 0 \tag{17.53}$$

has

$$\lambda^2 + 4\lambda + 4 = 0 \tag{17.54}$$

as its characteristic equation. The roots of the characteristic equation are given by

$$\begin{aligned} \lambda &= \tfrac{-4\pm\sqrt{16-16}}{2} \\ &= -2 \quad . \end{aligned} \tag{17.55}$$

Thus we have a double root and the general solution is

$$y(x) = c_1 e^{-2x} + c_2 x e^{-2x} \quad . \tag{17.56}$$

EXAMPLE 17.3. The differential equation

$$y'' + y' + y = 0 \tag{17.57}$$

has

$$\lambda^2 + \lambda + 1 = 0 \tag{17.58}$$

as its characteristic equation. The roots of the characteristic equation are

$$\begin{aligned} \lambda &= \tfrac{-1\pm\sqrt{1-4}}{2} \\ &= -\tfrac{1}{2} \pm i\tfrac{\sqrt{3}}{2} \end{aligned} \tag{17.59}$$

and so the general solution is

$$y(x) = c_1 e^{-\frac{1}{2}x}\cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 e^{-\frac{1}{2}x}\sin\left(\frac{\sqrt{3}}{2}x\right) \quad . \tag{17.60}$$