LECTURE 16

Second Order Linear Equations with Constant Coefficients

We shall now begin to investigate how to actually solve linear ODE's of degree 2. We shall begin with differential equations of a particularly simple type; equations of the form

$$y'' + py' + qy = 0 \tag{16.1}$$

where p and q are constant.

A clue as to how one might construct a solution to (16.1) comes from the observation that (16.1) implies that y'', y' and y are related to one another py multiplicative constants. There is one class of functions for which is certainly true: the exponential functions; i.e., functions of the form

$$y(x) = e^{\lambda x} \quad . \tag{16.2}$$

We will therefore look for solutions of (16.1) having the form (16.2).

Plugging (16.2) into (16.1) yields

$$0 = \lambda^2 e^{\lambda x} + p\lambda e^{\lambda x} + q e^{\lambda x} = \left(\lambda^2 + p\lambda + q\right) e^{\lambda x} \quad . \tag{16.3}$$

Since the exponential function $e^{\lambda x}$ never vanishes we must have

$$\lambda^2 + p\lambda + q = 0 \quad . \tag{16.4}$$

Equation (16.4) is called the **characteristic equation** for (16.1) since for any λ satisfying (16.4) we will have a solution $y(x) = e^{\lambda x}$ of (16.1).

Now because (16.4) is a quadratic equation we can employ the Quadratic Formula to find all of its roots:

$$\lambda^2 + p\lambda + q = 0 \qquad \Rightarrow \qquad \lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2} \quad . \tag{16.5}$$

Note that a root λ of (16.4) need not be a real number. Indeed, if $p^2 - 4q < 0$, then in order to compute λ via the Quadratic Formula we have to take the square root of a negative number and that forces us into the realm of complex numbers. We shall postpone temporarily the case when a root λ is complex and first discuss the case when the roots of (16.4) are all real. This requires $p^2 - 4q \geq 0$.

If $p^2 - 4q$ is positive, $\sqrt{p^2 - 4q}$ is a positive real number and

$$\begin{aligned} \lambda_+ &= \tfrac{-p+\sqrt{p^2-4q}}{2} \\ \lambda_- &= \tfrac{-p-\sqrt{p^2-4q}}{2} \end{aligned} \tag{16.6}$$

are distinct real roots of (16.4). Thus,

$$\begin{aligned} y_1(x) &= e^{\lambda_+ x} \\ y_2(x) &= e^{\lambda_- x} \end{aligned} \tag{16.7}$$

will both be solutions of (16.1). If we multiply these solutions py arbitrary constants c_1 and c_2, the resulting functions will still be solutions of (16.1). In fact, we can take arbitrary linear combinations of $y_1(x)$ and $y_2(x)$; sy,

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \tag{16.8}$$

and the result will still be a solution of (16.1).

For suppose $y_1(x)$ and $y_2(x)$ are the two solutions of (16.1) given above. Then if we set

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \tag{16.9}$$

we have

$$\begin{aligned} y'' + py' + qy &= a\frac{d^2}{dx^2}(c_1y_1 + c_2y_2) + b\frac{d}{dx}(c_1y_1 + c_2y_2) \\ &\qquad + c(c_1y_1 + c_2y_2) \\ &= c_1(y_1'' + py_1' + qy_1) + c_2(y_2'' + py_2' + qy_2) \\ &= c_1 \cdot 0 + c_2 \cdot 0 \\ &= 0 \end{aligned}$$

Example

$$y'' + 3y' + 2y = 0 \quad . \tag{16.10}$$

Setting

$$y(x) = e^{\lambda x} \tag{16.11}$$

and plugging into the differential equation we get

$$\begin{aligned} 0 &= \lambda^2 e^{\lambda x} + 3\lambda e^{\lambda x} + 2e^{\lambda x} \\ &= e^{\lambda x}\left(\lambda^2 + 3\lambda + 2\right) \\ &= e^{\lambda x}(\lambda + 1)(\lambda + 2) \end{aligned}$$

Since $e^{\lambda x}$ never vanishes (for any finite x), we must have

$$\lambda = -1 \quad \text{or} \quad \lambda = -2 \quad . \tag{16.12}$$

We thus find two distinct solutions

$$\begin{aligned} y_1(x) &= e^{-x} \\ y_2(x) &= e^{-2x} \quad . \end{aligned} \tag{16.13}$$

The general solution is thus

$$y(x) = c_1 e^{-x} + c_2 e^{-2x} \quad . \tag{16.14}$$