LECTURE 13

Change of Variable

We will now discuss one last technique for solving non-linear first order differential equations. Like the case of separable and exact equations, to employ this technique only works is special cases.

1. Homogeneous Equations of Degree Zero

The special form to be considered here is when the given differential equation

$$y' = f(x,y) \tag{13.1}$$

can be expressed as

$$y' = F\left(\frac{y}{x}\right) \quad . \tag{13.2}$$

In such a case, we say the differential equation (13.1) is **homogeneous of degree zero**.

EXAMPLE 13.1.

$$y' = \frac{y^2+2xy}{x^2} = \left(\frac{y}{x}\right)^2 + 2\left(\frac{y}{x}\right)^{-1} = F\left(\frac{y}{x}\right) \text{ when } F(v) \equiv v^2 + \frac{2}{v} \tag{13.3}$$

EXAMPLE 13.2.

$$y' = \ln(x) - \ln(y) = -ln\left(\frac{y}{x}\right) = F\left(\frac{y}{x}\right) \text{ when } F(v) \equiv -\ln(v) \tag{13.4}$$

EXAMPLE 13.3.

$$y' = \frac{x+y}{y-x} = \frac{1+\left(\frac{y}{x}\right)}{\left(\frac{y}{x}\right)-1} = F\left(\frac{y}{x}\right) \text{ when } F(v) \equiv \frac{1+v}{v-1} \tag{13.5}$$

To solve such equations we introduce a new variable which we will denote by v, to represent the ratio of y to x:

$$v = \frac{y}{x} \quad . \tag{13.6}$$

We then have

$$\frac{dv}{dx} = -\frac{y}{x^2} + \frac{1}{x}\frac{dy}{dx} \tag{13.7}$$

or

$$\frac{dy}{dx} = x\left(\frac{y}{x^2} + \frac{dv}{dx}\right) = v + x\frac{dv}{dx} \quad . \tag{13.8}$$

Hence equation (13.2) becomes

$$x\frac{dv}{dx} + v = F(v) \tag{13.9}$$

or

$$xdv = (F(v) - v)\,dx \tag{13.10}$$

or

$$\frac{dx}{x} = \frac{dv}{F(v)-v} \quad . \tag{13.11}$$

Integrating both sides of this equation yields

$$\ln|x| = \int \frac{dv}{F(v)-v} + C \quad . \tag{13.12}$$

Suppose now that the integral on the right hand side has been carried out so

$$H(v) = \int \frac{dv}{F(v)-v} \tag{13.13}$$

is some explicit function of v. Equation (13.12) becomes

$$\ln|x| - H(v) = C \tag{13.14}$$

or

$$\ln|x| - H\left(\frac{y}{x}\right) = C \quad . \tag{13.15}$$

We now have an equation specifying y as an implicit function of x. An explicit solution of the differential equation (13.1) is obtained whenever equation (13.15) can be solved for y in terms of x and C.

In summary, the change of variables $y \to v$ allows us to transform a differential equation of the form (13.2) to a separable differential equation (13.11) which can be solved by integrating both sides of (13.11) and then solving for v in terms of x and then substituting $\frac{y}{x}$ for v and solving for y in terms of x.

Example 13.4.

$$y' = \frac{x+y}{x} \tag{13.16}$$

This equation is homogeneous, since we can re-write it as

$$y' = 1 + \frac{y}{x} \quad . \tag{13.17}$$

We thus take

$$F(v) = 1 + v \tag{13.18}$$

and try to solve

$$\frac{dx}{x} = \frac{dv}{F(v)-v} = \frac{dv}{1+v-v} = dv \quad . \tag{13.19}$$

Integrating both sides yields

$$\ln(x) = v + C = \frac{y}{x} + C \tag{13.20}$$

or

$$y = x\left(\ln(x) - C\right) \quad . \tag{13.21}$$

2. Other Substitutions

The substitution method can also be applied in other situations; however, in such cases there usually isn't a litmus test that will tell you whether or not a given substitution will help solve the differential equation – instead one has to resort to trial and error in order to find an appropriate substitution. There is at least a guiding principle though: you want to make a substitution that will simply the differential equation. Here are some examples.

Example 13.5.

$$\frac{dy}{dx} = (x+y+1)^2 + 3 \tag{13.22}$$

In order to simplify the quadratic expression on the right hand side we'll try the following substitution:

$$z = x + y + 1 \quad \Rightarrow \quad \frac{dz}{dx} = 1 + \frac{dy}{dx} + 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{dz}{dx} - 1$$

Substituting z for $x+y+1$ on the right hand side of (13.22) and $z'-1$ for y' on the left hand side of (13.22) we obtain the following equivalent differential equation:

$$\frac{dz}{dx} - 1 = z^2 + 3$$

or

$$\frac{dz}{dx} = z^2 + 4$$

or

$$\frac{dz}{z^2+4} = dx$$

This equation is separable, and integrating both sides yields

$$\frac{1}{2}\arctan\frac{1}{2}z = \int \frac{dz}{z^2+4} = \int dx + C = x + C$$

or

$$\arctan\left(\frac{z}{2}\right) = 2x + C'$$

or

$$z = 2\tan\left(2x + C'\right)$$

or, recalling that $z = x + y + 1$,

$$x + y + 1 = 2\tan\left(2x + C'\right)$$

or, solving for y,

$$y = 2\tan\left(2x + C'\right) - x - 1$$