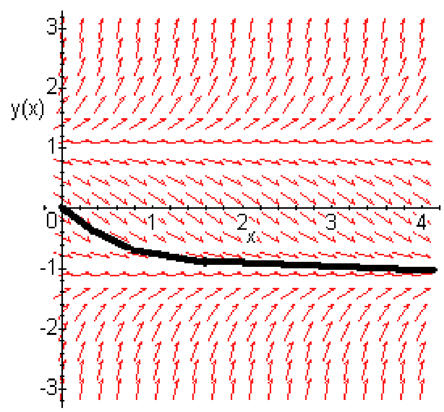
Sample Exam

Math 2233.003
FIRST EXAM
10:30 – 11:20 am, October 1, 1997

Name:__________________________________

1. Consider the plot below of the direction field for the differential equation $y' = (y-1)(y+1)$.



(a) (5 pts) Sketch the solution curve satisfying $y(0) = 0$.

(b) (5 pts) Suppose $y(x)$ is a solution satisfying $y(0) = -2$. What can you say about the asymptotic behavior of $y(x)$ as $x \to \infty$?

- The solution curves are have positive slope and so are increasing for all $y < -1$. However, at $y = -1$, the slope must be zero. Therefore, a solution satisfying $y(0) = -2$ will increase but asymptotically approach the line $y = -1$ as $x \to \infty$.

(c) (5 pts) Suppose $y(x)$ is a solution satisfying $y(1) = 0.5$. What can you say about the asymptotic behavior of $y(x)$ as $x \to \infty$?

- The solution curves are have negative slope and so are decreasing for all $-1 < y < 1$. However, at $y = -1$, the slope must be zero. Therefore, a solution satisfying $y(0) = -2$ will decrease but asymptotically approach the line $y = -1$ as $x \to \infty$.

2. (15 pts) Consider the following nonlinear first order ODE: $y' = y^2$ and suppose $y(x)$ is the solution satisfying $y(1) = 2$. Use the Euler method with $n = 3$ and $\Delta x = 0.1$ to estimate $y(1.3)$.

- Set

$$\begin{aligned} x_0 &= 1 \\ y_0 &= 2 \\ \Delta x &= 0.1 \end{aligned}$$

The slope of the solution passing throught the point (1,2) will be

$$m_0 = \left.\frac{dy}{dx}\right|_{(1,2)} = \left.y^2\right|_{(1,2)} = 2^2 = 4$$

Therefore we take the next point of the solution curve to be

$$\begin{aligned} x_1 &= x_0 + \Delta x = 1.1 \\ y_1 &= y_0 + m_0 \Delta x = 2 + (4)(0.1) = 2.4 \end{aligned}$$

The slope of the solution curve at this point (1.1,2.4) must then be

$$m_1 = \left.\frac{dy}{dx}\right|_{(1.1,2.4)} = \left.y^2\right|_{(1.1,2.4)} = (2.4)^2 = 5.76$$

so

$$\begin{aligned} x_2 &= x_1 + \Delta x = 1.2 \\ y_2 &= y_1 + m_1 \Delta x = 2.4 + (5.76)(0.1) = 2.976 \end{aligned}$$

Continuing, we calculate the slope at (1.2,2.976) to be

$$m_2 = \left.\frac{dy}{dx}\right|_{(1.2,2.976)} = \left.y^2\right|_{(1.2,2.976)} = (2.976)^2 = 8.8566$$

and so

$$\begin{aligned} x_3 &= x_2 + \Delta x = 1.3 \\ y_3 &= y_2 + m_2 \Delta x = 2.976 + (8.8566)(0.1) = 3.8617 \end{aligned}$$

So

$$y(1.3) = y_3 = 3.8617$$

3. (15 pts) Consider the following nonlinear first order ODE: $y' = x\cos(y)$. Write down the first four terms of the Taylor expansion of the solution satisfying $y(0) = 0$ about $x = 0$ (i.e. the terms up to order x^3).

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$$\begin{aligned} y'(x) &= x\cos(y) \\ y''(x) &= \cos(y) - x\sin(y)y'(x) \\ y'''(x) &= -\sin(y)y'(x) - \sin(y)y'(x) - x\cos(y)\left(y'(x)\right)^2 - x\sin(y)y''(x) \end{aligned}$$

Since $y(0) = 0$, we then have

$$\begin{aligned} y'(0) &= 0 \cdot 1 = 0 \\ y''(0) &= 1 - 0 \cdot 0 \cdot 1 = 1 \\ y'''(0) &= -0 \cdot 0 - 0 \cdot 0 - 0 \cdot 1 \cdot (0)^2 - 0 \cdot 0 \cdot 1 = 0 \end{aligned}$$

Hence

$$\begin{aligned} y(x) &= y(0) + y'(0)x + \frac{1}{2!}y''(0)x^2 + \frac{1}{3!}y'''(0)x^3 + \cdots \\ &= \frac{1}{2}x^2 + \cdots \end{aligned}$$

4. (20 pts) Find an explicit solution of the following (separable) initial value problem.

$$2x + \frac{1}{y} y' = 0 \qquad , \qquad y(1) = 1$$

- $$\frac{dy}{y} = -2x\,dx$$

  $$\ln|y| = \int \frac{dy}{y} = \int -2x\,dx + C = -x^2 + C$$

  When $x = 1$, $y = 1$, so we must have

  $$0 = \ln|1| = -1^2 + C$$

  or $C = 1$. Hence

  $$\ln|y| = -x^2 + 1$$

  or

  $$y = e^{1-x^2}$$

5. (15 pts) Solve the following initial value problem

$$y' - \frac{2}{x} y = x^3 \qquad , \qquad y(1) = 1$$

- $$p(x) = -\frac{2}{x}$$
  $$g(x) = x^3$$

  $$\mu(x) = \exp\left[\int p(x)dx\right] = \exp\left[-\int \frac{2dx}{x}\right] = \exp\left[-2\ln|x|\right] = \exp\left[\ln|x^{-2}|\right] = x^{-2}$$

  $$\begin{aligned} y(x) &= \frac{1}{\mu(x)} \int \mu(x) g(x) dx + \frac{C}{\mu(x)} \\ &= x^2 \int x^{-2} x^3 dx + Cx^2 \\ &= \frac{1}{2} x^4 + Cx^2 \end{aligned}$$

  We now plug into the initial condition

  6.

(a) (5 pts)Show that the following equation is not exact.

$$\left(3x^3 y + xy^2\right) + \left(2xy^2 + x^2 y\right) \frac{dy}{dx} = 0$$

- $$\begin{aligned} M &= 3x^3 y + xy^2 \;\Rightarrow\; \frac{\partial M}{\partial y} = 3x^3 + 2xy \\ N &= 2xy^2 + x^2 y \;\Rightarrow\; \frac{\partial N}{\partial x} = 2y^2 + 2xy \end{aligned}$$

  Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ the differential equation is not exact.

(b) (5 pts) Show that $\mu(x, y) = x^{-1} y^{-1}$ is an integrating factor for the equation in Part (a).

- Multiplying the differential equation by $\mu(x,y)$ we obtain

$$\frac{1}{xy}\left((3x^3y+xy^2)+\left(2xy^2+x^2y\right)\frac{dy}{dx}\right)=0$$

or

$$\left(3x^2+y\right)+(2y+x)\frac{dy}{dx}=0.$$

For this equation

$$\begin{aligned} M &= 3x^2+y \;\Rightarrow\; \frac{\partial M}{\partial y}=1 \\ N &= 2y+x \;\Rightarrow\; \frac{\partial N}{\partial x}=1 \end{aligned}$$

and so the new equation is exact $\left(\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}\right)$.

(c) (10 pts) Use the integrating factor in Part (b) to find the general solution of the differential equation in Part (a).

- Since

$$\left(3x^2+y\right)+(2y+x)\frac{dy}{dx}=0$$

is exact there must exist an equivalent algebraic equation of the form

$$\phi(x,y)=C$$

with the function $\phi(x,y)$ satisfying

$$\begin{aligned} \frac{\partial\phi}{\partial x} &= M=3x^2+y \\ \frac{\partial\phi}{\partial y} &= N=2y+x \end{aligned}$$

Un-doing the partial derivatives in the two equations above yields the following two 'guesses' for $\phi(x,y)$.

$$\begin{aligned} \phi(x,y) &= \int\frac{\partial\phi}{\partial x}\partial x+H_1(y)=\int\left(3x^2+y\right)\partial x+H_1(y)=x^3+xy+H_1(y) \\ \phi(x,y) &= \int\frac{\partial\phi}{\partial y}\partial y+H_2(x)=\int(2y+x)\partial y+H_2(x)=y^2+xy+H_2(x) \end{aligned}$$

Comparing these two expressions for $\phi(x,y)$, we see we must take $H_1(y)=y^2$, $H_2(x)=x^3$, and $\phi(x,y)=x^3+xy+y^2$. Hence our original differential equation is equivalent to the following algebraic equation:

$$x^3+xy+y^2=C.$$

Applying the quadratic formula to solve for y we obtain

$$y(x)=\frac{-x\pm\sqrt{x^2-4(x^3-C)}}{2}$$