LECTURE 9

Constants of Integration and Initial Conditions

Recall from our last lecture that the general solution of a first order linear differential equation

$$y' + p(x)y = g(x) \tag{9.1}$$

has the form

$$y(x) = \frac{1}{\mu(x)} \int^x \mu(x')g(x')dx' + \frac{C}{\mu(x)} \tag{9.2}$$

where

$$\mu(x) = \exp\left(\int^x p(x')dx'\right)$$

where C is an arbitrary constant.

In physical applications, one generally has the following situation. First, a differential equation is capable of describing a particular class of phenomena (such as the trajectory of a ball thrown up in the air) is derived from general theoretical principles (e.g., Newton's 2^{nd} Law of Motion). Then particular instances of this class of phenomena are prescribed in terms of their initial conditions (e.g., the initial position and the initial velocity of the ball). Thus, in applications one tends to look for simultaneous solutions of a differential equation and a certain set of initial conditions. Such problems are referred to as **initial value problems**.

EXAMPLE 9.1. Solve the following initial value problem:

$$\begin{aligned} y' + 2y &= xe^{-2x} \\ y(1) &= 0 \quad . \end{aligned}$$

Step 1. Solution of the differential equation.

Plugging into our general formula we quickly find

$$\begin{aligned} \mu(x) &= \exp\left[\int^x 2dx\right] = e^{2x} \\ y(x) &= \tfrac{1}{e^{2x}}\left[\int^x \left(e^{2x}\right)\left(xe^{-2x}\right)dx\right] + \tfrac{C}{e^{2x}} \\ &= \tfrac{x^2e^{-2x}}{2} + Ce^{-2x} \end{aligned}$$

Step 2. Plugging the result from Step 1 into the initial condition equation to fix the constant C.

$$\begin{aligned} 0 &= y(1) \\ &= \tfrac{1}{2} + C \\ \Rightarrow \quad C &= -\tfrac{1}{2} \\ \Rightarrow \quad y(x) &= \tfrac{x^2e^{-2x}}{2} - \tfrac{1}{2}e^{-2x} \\ &= \tfrac{1}{2}e^{-2x}\left(x^2 - 1\right) \quad . \end{aligned}$$

EXAMPLE 9.2. Find a solution to

$$y' + \frac{2}{x}y = \frac{\cos(x)}{x^2} \quad ,$$

on the interval $(0, +\infty)$ that satisfies the initial condition

$$y(\pi) = 0 \quad .$$

Since $p(x) = \frac{2}{x}$, one quickly calculates, exactly as in the example of the previous lecture, that

$$\mu(x) = \exp\left(\int^x \frac{2}{x}\,dx'\right) = x^2 \quad ,$$

and so

$$\begin{aligned} y(x) &= \frac{1}{x^2}\int x^2\left(\frac{\cos(x)}{x^2}\right)dx + \frac{C}{x^2} \\ &= \frac{\sin(x)}{x^2} + \frac{C}{x^2} \quad . \end{aligned}$$

We will now fix the value of the constant C by imposing the initial condition $y(\pi) = 0$.

$$0 = y(\pi) = \frac{\sin(\pi) + C}{x^2} = \frac{C}{x^2} \quad \Rightarrow \quad C = 0 \quad .$$

So

$$y(x) = \frac{\sin(x)}{x^2} \quad .$$

There is an even more expedient way of solving the initial value problem; one in which are initial conditions are inserted directly into the formula for y(x).

Recall that the function $\mu(x)$ used in the formula for $y(x)$ was defined as the exponential of the anti-derivative of $p(x)$:

$$\mu(x) = \exp\left(\int^x p(x')\,dx\right) \quad .$$

Since anti-derivatives are only determined up to the addition of an arbitrary constant, one is free to add to the term inside the parentheses an arbitrary number. And one particular way of doing this is by introducing a lower limit to the integration. Therefore, let us set

$$\mu_o(x) = \exp\left(\int_{x_o}^x p(x')\,dx\right) \quad .$$

Our claim is that if $\mu_o(x)$ is chosen in this way, then

$$y(x) = \frac{1}{\mu_o(x)}\int_{x_o}^x \mu_o(x')g(x')\,dx' \quad + \quad \frac{y_o}{\mu_o(x)}$$

is the unique solution of the initial value problem

$$\begin{aligned} y' + p(x)y &= g(x) \quad , \\ y(x_o) &= y_o \quad . \end{aligned}$$

We first show that

$$\lim_{x \to x_o} \mu_o(x) = 1 \quad .$$

This follows from the following observation:

$$\lim_{x \to x_o}\int_{x_0}^x p(x')\,dx' \leq \lim_{x \to x_o} (\textit{area under curve } y = p(x) \textit{ between } x_o \textit{ and } x) = 0$$

so,

$$\lim_{x \to x_o} \mu_o(x) = \exp\left(\lim_{x \to x_o} \int_{x_0}^{x} p(x')\,dx'\right) = e^0 = 1 \quad .$$

Thus, when we write the general solution of (9.1) as

$$y(x) = \frac{1}{\mu_o(x)} \int_{x_o}^{x} \mu_o(x') g(x')\,dx + \frac{C}{\mu_o(x)} \tag{9.3}$$

and plug into the initial value equation

$$y(x_o) = y_o \quad ,$$

we get

$$y_o = \frac{1}{1} \cdot 0 + \frac{C}{1} = C \quad .$$

(Comparing (9.3) with (9.2), notice that we have also added a lower limit to the integral of $\mu(x)g(x)$. The $y(x)$ so defined is still a solution of the differential equation (9.1); the effect of adding the lower limit is equivalent to a harmless change in the arbitrary constant C.)

Thus, the unique solution to the initial value problem

$$\begin{aligned} y' + p(x)y &= g(x) \\ y(x_o) &= y_o \end{aligned}$$

is given by the formula

$$\begin{aligned} & y(x) = \tfrac{1}{\mu_o(x)} \textstyle\int_{x_o}^{x} \mu_o(x') g(x') dx' + \tfrac{y_o}{\mu_o(x)} \\ \textit{where} \quad & \\ & \mu_o(x) = \exp\left(\textstyle\int_{x_o}^{x} p(x') dx'\right) \quad . \end{aligned} \tag{9.4}$$

Example 9.3.

$$y' + \cot(x) y = 3\csc(x) \qquad , \qquad \frac{\pi}{2} \le x < \pi \quad .$$
$$y\left(\frac{\pi}{2}\right) = 1$$

We first calculate $\mu_o(x)$.

$$\begin{aligned} \mu(x) &= \exp\left(\int_{\frac{\pi}{2}}^{x} \cot(x')\,dx'\right) \\ &= \exp\left(\ln\left(|\sin(x)| - \ln\left(|\sin(\frac{\pi}{2})|\right)\right)\right) \\ &= |\sin(x)| \quad . \\ &= \sin(x) \qquad \text{when } \frac{\pi}{2} \le x < \pi \end{aligned}$$

We can now apply formula (9.4):

$$\begin{aligned} y(x) &= \frac{1}{\sin(x)} \int_{\frac{\pi}{2}}^{x} 2\sin(x')\csc(x')\,dx' + \frac{1}{\sin(x)} \\ &= \frac{1}{\sin(x)} \int_{\frac{\pi}{2}}^{x} 2\,dx' + \frac{1}{\sin(x)} \\ &= \frac{1}{\sin(x)}\left[2x - \pi + 1\right] \end{aligned}$$