LECTURE 8

# First Order Linear Differential Equations

We now turn our attention to the problem of constructing analytic solutions of differential equations; that is to say, solutions that can be expressed in terms of elementary functions (or formulae). We consider first the case of first order linear differential equations.

## 1. Linear vs Non-Linear Differential Equations

An ordinary or partial differential equation is said to be **linear** if it is linear in the "unknowns" $f$ , $\frac{df}{dx}$ , $\frac{d^2f}{dx^2}$, etc.. Thus, a general, linear, ordinary, $n^{th}$ order, differential equation would be one of the form

$$a_n(x)\frac{d^nf}{dx^n}(x) + a_{n-1}(x)\frac{d^{n-1}f}{dx^{n-1}}(x) + \cdots + a_1(x)\frac{df}{dx}(x) + f(x) = g(x) \quad .$$

It is important to note that the functions $a_n(x), \ldots, a_1(x), g(x)$ need not be linear functions of $x$. The following two examples should convey the general idea.

Example 8.1.

$$x^2\frac{\partial f}{\partial x} + z\frac{\partial^2 f}{\partial y^2} = e^{zxy}$$

is a $2^{nd}$ order, linear, partial, differential equation.

Example 8.2.

$$\frac{d^3f}{dx^3} + x^2\frac{df}{dx} + f^2 = 1$$

is a non-linear, ordinary, differential equation of order 3. The equation is non-linear arises because of the presence of the term $f^2$ which is a quadratic function of the unknown function $f$.

## 2. Solving First Order Linear ODEs

A *linear first order ordinary differential equation* is a differential equation of the form

$$a(x)y' + b(x)y = c(x) \quad . \tag{8.1}$$

Here $y$ represents the unknown function, $y'$ its derivative with respect to the variable $x$, and $a(x), b(c)$ and $c(x)$ are certain prescribed functions of $x$ (the precise functional form of $a(x)$, $b(x)$, and $c(x)$ will fixed in any specific example). So long as $a(x) \neq 0$, this equation is equivalent to a differential equation of the form

$$y' + p(x)y = g(x) \tag{8.2}$$

where

$$\begin{aligned} p(x) &= \frac{b(x)}{a(x)}, \\ g(x) &= \frac{c(x)}{a(x)}. \end{aligned}$$

We shall refer to a differential equation (8.2) as the **standard form** of differential equation (8.1). (In general, we shall say that an ordinary linear differential equation is in **standard form** when the coefficient of the highest derivative is 1.)

Our goal now is to develop a formula for the general solution of (8.2). To accomplish this goal, we shall first construct solutions for several special cases. Then with the knowledge gained from these simpler examples, we will develop a general formula for the solution of **any** differential equation of the form (8.2).

*Case (i)* $p(x) = 0$, $g(x) =$ arbtrary function.

In this case, we have

$$\frac{dy}{dx} = g(x) \quad ,$$

and so we are looking for a function whose derivative is $g(x)$. Applying the Fundamental Theorem of Calculus (integral $\Leftrightarrow$ anti-derivative), we have

$$y(x) = \int \frac{dy}{dx}\, dx = \int g(x)\, dx = \int^x g(x)\, dx \quad + \quad C \quad ,$$

where $C$ is an arbitrary constant of integration.

Example 8.3.

$$y' = 3\cos(4x)$$

$$\begin{aligned} \Rightarrow \quad y(x) &= \int^x 3\cos(4x)\, dx \quad + \quad C \\ &= \frac{3}{4}\sin(4x) + C \end{aligned}$$

*Case (ii)*: $g(x) = 0$, $p(x) =$ arbitrary function.

In this case we are trying to solve a differential equation of the form

$$y' + p(x)y = 0 \quad . \tag{8.3}$$

To construct a solution, we first re-write (8.3) as an equation involving differentials

$$\begin{aligned} \frac{dy}{dx} + p(x)y &= 0 \\ \Rightarrow \quad \frac{dy}{y} &= -p(x)dx \end{aligned}$$

Integrating both sides of the latter equation (the left hand side with respect to $y$ and the right hand side with respect to $x$) yields

$$\ln(y) = -\int^x p(x')\, dx' \quad + \quad C$$

or, exponentiating both sides

$$y = \exp\left[-\int^x p(x')\, dx' \quad + \quad C\right]$$

$$
\begin{aligned}
y &= \exp\left[-\int^x p(x')\,dx' \quad + \quad C\right] \\
&= e^C \exp\left[-\int^x p(x')\,dx'\right] \\
&= A \exp\left[-\int^x p(x')\,dx'\right]
\end{aligned}
$$

In the last step we have simply replaced the constant $e^C$, which is arbitrary since $C$ is arbitrary, by another arbitrary constant $A$. There is nothing tricky here; the point is that in the general solution the numerical factor in front of the exponential function is arbitrary and so rather than writing this factor as $e^C$ we use the simpler form $A$. Thus, the general solution of

$$y' + p(x) = 0$$

is

$$y = A \exp\left[-\int^x p(x')\,dx'\right] \quad .$$

We note that in both Cases (i) and (ii), we constructed a solution by carrying out a single integration, and in doing so an arbitrary parameter (due to a constant of integration) was introduced. This is typical of first order differential equations. Indeed, a general solution to an $n^{th}$-order differential equation will involve $n$ arbitrary parameters. We shall see latter that in physical applications these arbitrary constants correspond to initial conditions.

*Case (iii)*: $g(x) \neq 0$, $p(x) = a$, a constant.

In this case we have

$$\frac{dy}{dx} + ay = g(x) \quad .$$

To solve this equation we employ a trick. (This will not be the last trick you see in this course.) Let's multiply both sides of this equation by $e^{ax}$:

$$e^{ax}y' + ae^{ax}y = e^{ax}g(x).$$

Noticing that the right hand side is $\frac{d}{dx}\left(e^{ax}y\right)$ (via the product rule for differentiation) we have, equivalently,

$$\frac{d}{dx}\left(e^{ax}y\right) = e^{ax}g(x) \quad .$$

We now take anti-derivatives of both sides to get

$$e^{ax}y = \int^x e^{ax'}g(x')\,dx' \quad + \quad C$$

or

$$y(x) = \frac{1}{e^{ax}}\int^x e^{ax'}g(x')\,dx' \quad + \quad Ce^{-ax} \quad .$$

Example 8.4.

$$y' - 2y = x^2e^{2x}$$

This equation is of type (iii) with

$$
\begin{aligned}
p &= -2 \\
g(x) &= x^2e^{2x} \quad .
\end{aligned}
$$

So we multiply both sides by $e^{-2x}$ to get

$$\frac{d}{dx}\left(e^{-2x}y\right) = e^{2x}\left(y' - 2y\right) = e^{-2x}\left(x^2e^{2x}\right) = x^2$$

Integrating both sides with respect to $x$, and employing the Fundamental Theorem of Calculus on the left yields

$$e^{-2x}y = \frac{1}{3}x^3 + C$$

or

$$y = \frac{1}{3}x^3 e^{2x} + Ce^{2x} \quad .$$

Let us now confirm that this is a solution

$$\begin{aligned} y' &= x^2 e^{2x} + \frac{2}{3}x^3 e^{2x} + 2Ce^{2x} \\ -2y &= -\frac{2}{3}x^3 e^{2x} - 2Ce^{2x} \end{aligned}$$

so

$$y' - 2y = x^2 e^{2x}$$

□

## 3. The General Case

We are now prepared to handle the case of a general first order linear differential equation; i.e., differential equations of the form

$$y' + p(x)y = g(x) \tag{8.4}$$

with $p(x)$ and $g(x)$ are arbitrary functions of $x$.

**Note:** This case includes all the preceding cases.

We shall construct a solution of this equation in a manner similar to case when $p(x)$ is a constant; that is, we will try find a function $\mu(x)$ satisfying

$$\mu(x)\left(y' + p(x)y\right) = \frac{d}{dx}\left(\mu(x)y\right) \tag{8.5}$$

Multiplying (8.4) by $\mu(x)$, we could then obtain

$$\frac{d}{dx}\left(\mu(x)y\right) = \mu(x)g(x)$$

which when integrated yields

$$\mu(x)y = \int^x \mu(x')g(x')\,dx' + C$$

or

$$y = \frac{1}{\mu(x)}\int^x \mu(x')g(x')\,dx' \; + \; \frac{C}{\mu(x)} \tag{8.6}$$

It thus remains to find a suitable function $\mu(x)$; i.e., we need to find a function $\mu(x)$ so that

$$\frac{d}{dx}\left(\mu(x)y\right) = \mu(x)y' + \mu(x)yp(x)$$

This will certainly be true if

$$\frac{d}{dx}\mu(x) = p(x)\mu(x) \quad . \tag{8.7}$$

Thus, we have to solve another first order, linear, differential equation of type (iii). As before we re-write (8.7) in terms of differentials to get

$$\frac{d\mu}{\mu} = p(x)dx \quad ,$$

and then integrate both sides; yielding

$$\ln(\mu) = \int^x p(x')\, dx' + A \quad .$$

Exponentiating both sides of this relation yields

$$\mu = \exp\left(\int^x p(x')dx' + A\right)$$

$$\begin{aligned} \mu &= \exp\left(\int^x p(x')dx' + A\right) \\ &= A' \exp\left(\int^x p(x')\, dx'\right) \end{aligned}$$

So a suitable function $\mu(x)$ is

$$\mu(x) = A' \exp\left(\int^x p(x')dx'\right)$$

Inserting this expression for $\mu(x)$ into our formula (14) for $y$ yields

$$y(x) = \frac{1}{A' \exp\left(\int^x p(x'')dx''\right)} \left[\int^x A' \exp\left(\int^{x'} p(x')dx''\right) g(x')dx' \quad + \quad C\right]$$

It is easily see that the constant $A'$ in the denominator is irrelevant to the final answer. This is because it can be canceled out by the $A'$ within the integral over $x'$, and it can be absorbed into the arbitrary constant $C$ in the second term. Thus, the general solution to a first order linear equation

$$y' + p(x)y = g(x)$$

is given by

$$\begin{aligned} y(x) &= \frac{1}{\mu(x)} \int^x \mu(x')g(x')dx' + \frac{C}{\mu(x)} \\ \mu(x) &= \exp\left(\int^x p(x')dx'\right) \end{aligned} \tag{8.8}$$

Example 8.5.

$$xy' + 2y = \sin(x) \tag{8.9}$$

Putting this equation in standard form requires we set

$$\begin{aligned} p(x) &= \tfrac{2}{x} \\ g(x) &= \tfrac{\sin(x)}{x} \end{aligned}$$

Now

$$\int p(x)\, dx = \int \frac{2}{x} dx = 2\ln(x) = \ln\left(x^2\right),$$

so

$$\begin{aligned}\mu(x) &= \exp\left[\int^x p(x')\,dx'\right] \\ &= \exp\left[\ln\left(x^2\right)\right] \\ &= x^2\end{aligned}$$

Therefore,

$$\begin{aligned}y(x) &= \frac{1}{\mu(x)}\int^x \mu(x')g(x')\,dx' + \frac{C}{\mu(x)} \\ &= \frac{1}{x^2}\int^x (x')^2\frac{\sin(x')}{x'}\,dx' + \frac{C}{x^2} \\ &= \frac{1}{x^2}\int^x x'\sin(x')\,dx' + \frac{C}{x^2}\end{aligned}$$

Now

$$\int x\sin(x)\,dx$$

can be integrated by parts. Set

$$u = x \qquad , \qquad dv = \sin(x)dx$$

Then

$$du = dx \qquad , \qquad v = \int dv = -\cos(x)$$

and the integration by parts formula,

$$\int udv = uv - \int v\,du \quad ,$$

tells us that

$$\begin{aligned}\int x\sin(x)\,dx &= -x\cos(x) + \int \cos(x)\,dx \\ &= -x\cos(x) + \sin(x) \quad .\end{aligned}$$

Therefore, we have as a general solution of (8.9),

$$\begin{aligned}y(x) &= \frac{1}{x^2}\left(-x\cos(x) + \sin(x)\right) + \frac{C}{x^2} \\ &= \frac{1}{x^2}\sin(x) - \frac{1}{x}\cos(x) + \frac{C}{x^2} \quad .\end{aligned}$$

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