LECTURE 7

Separation of Variables

Suppose that a first order differential equation

$$y' = F(x, y) \tag{7.1}$$

can be written in the form

$$M(x) + N(y)\frac{dy}{dx} = 0 \quad . \tag{7.2}$$

Note that the first term depends only on x and the second term depends only on y and y'. In such a case, we say that the differential equation (7.1) is **separable**. Such differential equations can always be solved (at least implicitly).

To construct a solution of (7.2) we rewrite (7.2) as

$$M(x)dx = -N(y)dy \quad .$$

Integrating both sides of this relation yields

$$\int M(x)dx = -\int N(y)dy + C \quad . \tag{7.3}$$

The constant C is an arbitrary constant of integration. This equation can be used to establish y as an implicit function of x.

To see this, let us define functions $H_1(x)$ and $H_2(y)$ by

$$\begin{aligned} H_1(x) &= \int^x M(x')\,dx' \quad , \\ H_2(y) &= \int^y N(y')\,dy' \quad . \end{aligned}$$

Equation (7.3) is now equivalent to

$$H_1(x) = -H_2(y) + C \quad . \tag{7.4}$$

But now equation (7.4) expresses a purely algebraic relation between x and y. Solving (7.4) for y will then give us y as an explicit function of x.

Below is an argument that is a little more rigorous. If we set

$$\begin{aligned} H_1(x) &= \textstyle\int^x M(x')\,dx' \quad , \\ H_2(y) &= \textstyle\int^y N(y')\,dy' \end{aligned}$$

Then by the Fundamental Theorem of Calculus we have

$$\begin{aligned} \frac{dH_1}{dx} &= M(x) \quad , \\ \frac{dH_2}{dy} &= N(y) \quad . \end{aligned}$$

and so

$$M(x) + N(y)y' = 0$$

can be written

$$\begin{aligned} 0 &= \tfrac{dH_1}{dx} + \tfrac{dH_2}{dy}\tfrac{dy}{dx} \\ &= \tfrac{dH_1}{dx} + \tfrac{d}{dx}H_2\left(y(x)\right) \\ &= \tfrac{d}{dx}\left(H_1 + H_2\left(y(x)\right)\right) \end{aligned}$$

(Note that we have employed the "chain rule" in the second step.) But

$$\frac{d}{dx}\left(H_1(x) + H_2\left(y(x)\right)\right) = 0$$

implies that

$$H_1(x) + H_2\left(y\right) = C \quad , \quad \text{(some constant)}.$$

Solving this equation for y as a function of x and C will thus furnish us with a solution of (7.2).

In summary, the general solution of a nonlinear differential equation of the form

$$M(x) + N(y)\frac{dy}{dx} = 0$$

is constructed by first computing anti-derivatives $H_1(x)$, $H_2(y)$ of the functions the functions $M(x)$ and $N(y)$;

$$\begin{aligned} H_1(x) &= \textstyle\int^x M(x')\,dx' \\ H_2(y) &= \textstyle\int^y N(y')dy' \end{aligned}$$

and solving the equation

$$H_1(x) + H_2(y) = C$$

for y.

Example 7.1.

$$y' = \frac{y^2}{x} \tag{7.5}$$

After multiplying both sides by $\frac{x}{y^2}$, this equation can also be rewritten as

$$\frac{1}{x} = \frac{1}{y^2}\frac{dy}{dx} \quad ;$$

or

$$\frac{dx}{x} = \frac{dy}{y^2} \quad .$$

Integrating the left hand side with respect to x and the right hand side with respect to y yields

$$\ln|x| = \int\frac{dx}{x} = \int\frac{dy}{y^2} + C = -\frac{1}{y} + C$$

or

$$\frac{1}{y} = -\ln|x| + C$$

or

$$y(x) = \frac{1}{C - \ln|x|} \quad .$$

The equation above represents the general solution of (7.5). ∎