LECTURE 6

General Theory of First Order Ordinary Differential Equations

There are several basic questions one can ask about a differential equation.

1. **The Existence Problem.** Does a solution exist? How can one tell that a solution does exist?
2. **The Uniqueness Problem.** If a solution does exist, how unique is it? Are there other solutions? Is there a way to parameterize all the solutions of a given differential equation.
3. **The Solution Problem.** How does one actually solve (or construct a solution of) a given differential equation?

In the preceding lectures we have addressed only the third problem. That is, we have shown how to construct approximate solutions, but we have not discussed whether or not it is possible to find other solutions, or when our method for constructing a solution might fail. I now wish to present a more thorough discussion of these problems.

The questions of existence and uniqueness for the solutions of a first order linear differential equation is answered by the following fundamental theorem (which we shall not prove):

THEOREM 6.1. *Suppose $F(x,y)$ and $\frac{\partial F}{\partial y}$ are continuous functions on an open rectangle*

$$R = \{(x,y) \mid \alpha < x < \beta \ , \ \gamma < y < \delta \ \}$$

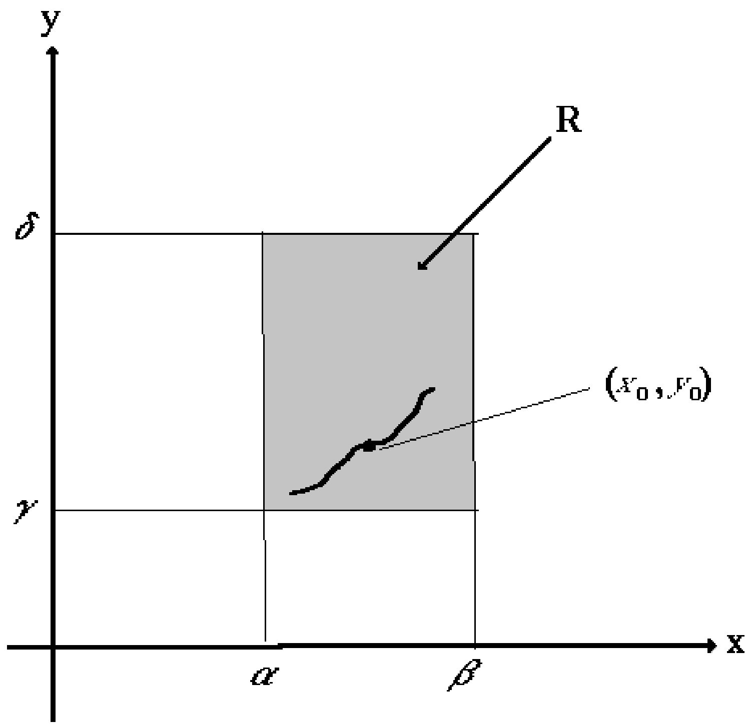
containing the point (x_o, y_o). Then there exists one and only one solution $y(x)$ of the differential equation

$$y' + p(x)y = g(x) \quad , \quad \forall\, x \in (\alpha, \beta) \tag{6.1}$$

defined on an open neighborhood $(x_o - h, x_o + h)$ of the point x_o satisfying the initial condition

$$y(x_o) = y_o. \tag{6.2}$$

Although the language of this theorem may seem a bit technical, the theorem really easy to apply and it yields a very strong result. To better understand this theorem, consider the figure below.



Simply put, the theorem tells us

1. If we want to know if there is a solution of (6.1) satisfying (6.2) all we have to do is check to see if the functions $F(x,y)$ and $\frac{\partial F}{\partial y}$ are continuous within some rectangle containing the point (x_o, y_o).
2. If we have one solution of (6.1) and (6.2) and both $F(x,y)$ and $\frac{\partial F}{\partial y}$ are continuous around (x_o, y_o), then we have all the solutions.

EXAMPLE 6.2. On what intervals can we expect unique solutions to

$$\frac{dy}{dx} = \frac{y^2}{1-x^2}.$$

In this example, we have

$$\begin{aligned} F(x,y) &= \frac{y^2}{1-x^2} \\ \frac{\partial F}{\partial y}(x,y) &= \frac{2y}{1-x^2} \end{aligned}$$

These are both continuous functions so long as we avoid the lines $x = \pm 1$. The Existence and Uniqueness Theorem tells that we can expect one and only one solution of

$$\begin{aligned} \frac{dy}{dx} &= \frac{y^2}{1-x^2} \\ y(x_o) &= y_o \end{aligned}$$

so long as $x_o \neq \pm 1$; that is to say so long as $x_o \in (-\infty, -1) \cup (-1, +1) \cup (+1, +\infty)$.

EXAMPLE 6.3. Consider the following non-linear initial value problem.

$$\begin{aligned} y' = \frac{1}{2}\left(-x + \sqrt{x^2+4y}\right) \quad , \\ y(2) = -1 \quad . \end{aligned}$$

We verify that the functions

$$\begin{aligned} y_1(x) &= 1 - x \\ y_2(x) &= -\frac{x^2}{4} \end{aligned}$$

are both solutions to this non-linear initial value problem.

(i)

$$\begin{aligned} \frac{1}{2}\left(-x + \sqrt{x^2 + 4y_1}\right) &= \frac{1}{2}\left(-x + \sqrt{x^2 + 4(1-x)}\right) \\ &= \frac{1}{2}\left(-x + \sqrt{(x-2)^2}\right) \\ &= \frac{1}{2}(-x + x - 2) \\ &= -1 \\ &= y_1' \end{aligned}$$

(ii)

$$\begin{aligned} \frac{1}{2}\left(-x + \sqrt{x^2 + 4y_2}\right) &= \frac{1}{2}\left(-x + \sqrt{x^2 - 4\frac{x^2}{4}}\right) \\ &= \frac{1}{2}(-x) \\ &= \frac{-x}{2} \\ &= \frac{-2x}{4} \\ &= y_2' \end{aligned}$$

This example does not contradict the theorem above because the function corresponding to the function $F(x,y)$ of the theorem, i.e.,

$$F(x,y) = \frac{1}{2}\left(-x + \sqrt{x^2 + 4y}\right)$$

does not have a continuous partial derivative $\frac{\partial F}{\partial y}$ at point $(x_o, y_o) = (2, -1)$: indeed,

$$\frac{\partial F}{\partial y}(x,y) = \frac{1}{\sqrt{x^2 + 4y}}$$

is undefined when $x = 2$ and $y = -1$.

1. Nonlinear First Order Ordinary Differential Equations

We now turn our attention to the problem of constructing solutions of nonlinear first order ODEs. It turns out that the odds of being able to construct a nice formula for the solution of

$$y' = F(x,y)$$

are very slim. Only in five special cases will we be able to make any progress.

1. **Separable Equations**
2. **Linear Equations**
3. **Exact Equations**
4. **Equations with Integrating Factors**

5. **Homogeneous Equations**

We shall begin a discussion of each of these cases now starting with Separable Equations.