LECTURE 5

Taylor Series Methods

In this lecture I shall describe one last general method that is available to use to find approximate solutions of a first order differential equation.

Recall that the n^{th} order Taylor expansion of a (smooth) function $f(x)$ about the point $x = x_o$ is the degree n polynomial defined by

$$
\begin{aligned}
(5.1) \qquad T_n(x) &= \sum_{i=0}^{n} \frac{1}{i!} f^{(i)}(x_o)\,(x - x_o)^i \\
&= f(x_o) + f'(x_o)\,(x - x_o) + \frac{1}{2} f''(x_o)\,(x - x_o)^2 + \frac{1}{6} f'''(x_o)\,(x - x_0)^3 + \cdots
\end{aligned}
$$

and that such expansions are extremely useful in that they can (for sufficiently small $|x - x_0|$) be used as approximate expressions for the original function f. Indeed, Taylor's theorem says

$$f(x) = T_n(x) + \mathcal{O}\left(|x - x_o|^{n+1}\right)$$

and that moreover

$$f(x) = \lim_{n\to\infty} T_n(x)$$

(so long as $f(x)$ is smooth).

Therefore, one way to get an approximate solution of a differential equations would be to figure out what its Taylor series looks like and this turns out to be a relatively easy thing to do.

Suppose $y(x)$ is a solution of

$$(5.2) \qquad y' = F(x, y)$$

satisfying the initial condition

$$(5.3) \qquad y(x_o) = y_o.$$

Since $x = x_o$ implies $y = y_o$, and because the differential equation tells us what $y'(x)$ must be given x and y, we can infer that

$$y'(x_o) = y_o.$$

Thus, we already know the first two terms of the Taylor expansion of $y(x)$:

$$
\begin{aligned}
y(x) &= y(x_o) + y'(x_o)(x - x_0) + \cdots \\
&= y_o + F(x_o, y_o)(x - x_o) + \cdots
\end{aligned}
$$

What about the higher order terms? To get the second order term we can differentiate the original differential equation with respect to x to get

$$\begin{aligned} y''(x) &= \frac{d}{dx}F(x,y(x)) \\ &= \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} \\ &= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y'(x) \\ &= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}F\left(x,y(x)\right) \end{aligned}$$

So

$$y''(x_o) = \left.\frac{\partial F}{\partial x}\right|_{\substack{x=x_o \\ y=y_o}} + \left.\frac{\partial F}{\partial y}F(x,y)\right|_{\substack{x=x_o \\ y=y_o}}$$

which after carrying out the partial differentiations and plugging in for x and y is just a number. And so we now have the second order term of the Taylor expansion of our solution $y(x)$ about $x = x_o$. To get the third order term, we can differentiate the differential equation again to obtain

$$y'''(x) = \frac{d^2}{dx^2}\left(\frac{dy}{dx}\right) = \frac{d^2}{dx^2}F\left(x,y(x)\right) \tag{5.4}$$

So

$$y'''\left(x_o\right) = \left.\frac{d^2}{dx^2}F\left(x,y(x)\right)\right|_{\substack{x=x_o \\ y=y_o}}$$

Let's now look at a specific example.

EXAMPLE 5.1. Find the first four terms of the Taylor expansion of the solution of

$$y' = x + y^2 \tag{5.5}$$

$$y(0) = 1 \tag{5.6}$$

about $x = x_o = 0$.

Suppose $y(x)$ is a solution of the differential equation (5.5). Its Taylor series about $x = 0$ is then

$$y(x) = y(0) + y'(0)(x-0) + \frac{1}{2!}y''(0)(x-0)^2 + \frac{1}{3!}y'''(0)(x-0)^3 + \cdots \tag{5.7}$$

$$= y(0) + y'(0)x + \frac{1}{2}y''(0)x^2 + \frac{1}{6}y'''(0)x^3 + \cdots \tag{5.8}$$

Now the initial condition (5.6) gives us a value f or the first term; namely, $y(0) = 1$. The differential equation gives us a value for the factor $y'(0)$ in the second term; for Equation (**??**) says

$$y'(x) = x + \left(y(x)\right)^2 \tag{5.9}$$

so in particular

$$y'(0) = 0 + \left(y(0)\right)^2 = 0 + 1^2 = 1.$$

To get the factor $y''(0)$ in the third term we differentiate Equation (5.9) with respect to x:

$$y''(x) = \frac{d}{dx}\left(x + \left(y(x)\right)^2\right) \tag{5.10}$$

$$= 1 + 2y(x)y'(x) \tag{5.11}$$

$$= 1 + 2y(x)\left(x + \left(y(x)\right)^2\right) \tag{5.12}$$

$$= 1 + 2xy(x) + 2\left(y(x)\right)^3 \tag{5.13}$$

(In passiing from the second line to the third we have again employed Equation (5.9).) At $x = 0$, we then have

$$\begin{aligned} y''(0) &= 1 + 2 \cdot 0 \cdot y(0) + 2\,(y(0))^3 \\ &= 1 + 0 + 2 \cdot 1^3 \\ &= 3 \end{aligned}$$

To get a number for $y'''(0)$, we differentiate Equation (5.13) to get

$$\begin{aligned} y'''(x) &= 0 + 2y(x) + 2xy'(x) + 6\,(y(x))^2\, y'(x) \\ &= 0 + 2y(x) + 2x\left(x + (y(x))^2\right) + 6\,(y(x))^2\left(x + (y(x))^2\right) \\ &= 2y(x) + 2x^2 + 2x\,(y(x))^2 + 6x\,(y(x))^2 + 6\,(y(x))^4 \end{aligned}$$

Evaluating this equation at $x = 0$ yields

$$\begin{aligned} y'''(0) &= 2y(0) + 2 \cdot 0^2 + 2 \cdot 0 \cdot (y(0))^2 + 6 \cdot 0 \cdot (y(0))^2 + 6\,(y(0))^4 \\ &= 2 \cdot 1 + 0 + 0 + 0 + 6 \cdot 1^4 \\ &= 2 + 6 \\ &= 8 \end{aligned}$$

Finally, we can plug the values we found for $y(0)$, $y'(0)$, $y''(0)$, and $y'''(0)$ into the right hand side of Equation (5.8) to get

$$\begin{aligned} y(x) &= 1 + 1 \cdot x + \frac{1}{2} \cdot 3 \cdot x^2 + \frac{1}{6} \cdot 8 \cdot x^3 + \cdots \\ &= 1 + x + \frac{3}{2}x^2 + \frac{4}{3}x^3 + \cdots \end{aligned}$$

which we would could then view as an approximate solution of the intial value problem, accurate to order x^4.