Hyperbolic Functions

The purpose of this lecture is to introduce you to some basic functions that are frequently used in engineering, physics, and a variety of other applications.

The good news is that they really pretty simple functions, especially when expressed in terms of the exponential function:

\[
\sinh(x) = \frac{e^x - e^{-x}}{2} \\
\cosh(x) = \frac{e^x + e^{-x}}{2}
\]

The other hyperbolic functions are obtained from these two in a manner that mimics exactly how the other trigonometric functions can be obtained from the sine and cosine functions:

<table>
<thead>
<tr>
<th>notation</th>
<th>name</th>
<th>“trig-like formula”</th>
<th>formula in terms of exponentials</th>
</tr>
</thead>
<tbody>
<tr>
<td>tanh(x)</td>
<td>hyperbolic tangent</td>
<td>( \frac{\sinh(x)}{\cosh(x)} )</td>
<td>( \frac{e^x - e^{-x}}{e^x + e^{-x}} )</td>
</tr>
<tr>
<td>coth(x)</td>
<td>hyperbolic cotangent</td>
<td>( \frac{\cosh(x)}{\sinh(x)} )</td>
<td>( \frac{e^x + e^{-x}}{e^x - e^{-x}} )</td>
</tr>
<tr>
<td>sech(x)</td>
<td>hyperbolic secant</td>
<td>( \frac{1}{\cosh(x)} )</td>
<td>( \frac{e^x + e^{-x}}{2} )</td>
</tr>
<tr>
<td>csch(x)</td>
<td>hyperbolic cosecant</td>
<td>( \frac{1}{\sinh(x)} )</td>
<td>( \frac{e^x - e^{-x}}{2} )</td>
</tr>
</tbody>
</table>

Not only do these functions have definitions similar to trigonometric functions, their derivatives are also quite similar - in fact, even a little bit simpler because you don’t have to worry so much about which formulas use minus signs. Let’s compute a few derivatives.

First of all

\[
\frac{d}{dx} \sinh(x) = \frac{1}{2} \frac{d}{dx} (e^x - e^{-x}) = \frac{1}{2} e^x - \frac{1}{2} e^{-x}
\]

Using

\[
\frac{d}{dx} e^{\lambda x} = \lambda e^{\lambda x}
\]

we obtain

\[
\frac{d}{dx} \sinh(x) = \frac{1}{2} e^x - \frac{1}{2} (-e^{-x}) = \frac{e^x + e^{-x}}{2} = \cosh(x)
\]

Similarly,

\[
\frac{d}{dx} \cosh(x) = \frac{1}{2} \frac{d}{dx} (e^x + e^{-x}) = \frac{1}{2} e^x + \frac{1}{2} e^{-x} = \frac{1}{2} e^x + \frac{1}{2} (-e^{-x}) = \frac{e^x - e^{-x}}{2} = \sinh(x)
\]

Before computing more derivatives let me show another way in which the hyperbolic functions act like the trigonometric functions: that is, that they also satisfy a number of simple identities. For example,

\[
\cosh^2(x) - \sinh^2(x) = 1
\]
To see this, let’s write the left hand side in terms of exponential functions:

\[
\cosh^2(x) - \sinh^2(x) = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2
\]

\[
= \frac{1}{4} (e^{2x} + 2 + e^{-2x}) - \frac{1}{4} (e^{2x} - 2 + e^{-2x})
\]

\[
= \frac{1}{4} (e^{2x} + 2 + e^{-2x} - e^{2x} - 2 - e^{-2x})
\]

\[
= \frac{4}{4} = 1
\]

To compute the derivative of \(\tanh(x)\), it is easiest to use its “trig-like” formula, the preceding results, and the Quotient Rule.

\[
\frac{d}{dx} \tanh(x) = \frac{d}{dx} \frac{\sinh(x)}{\cosh(x)} = \frac{\frac{d}{dx} \sinh(x) \cosh(x) - \sinh(x) \frac{d}{dx} \cosh(x)}{(\cosh(x))^2}
\]

\[
= \frac{\cosh(x) \cosh(x) - \sinh(x) \sinh(x)}{\cosh^2(x)}
\]

\[
= \frac{1}{\cosh^2(x)} = \text{sech}^2(x)
\]

Below we give a table of the derivatives of all the hyperbolic functions

\[
\begin{align*}
\frac{d}{dx} \sinh(x) &= \cosh(x) \\
\frac{d}{dx} \cosh(x) &= \sinh(x) \\
\frac{d}{dx} \tanh(x) &= \text{sech}^2(x) \\
\frac{d}{dx} \coth(x) &= -\text{csch}^2(x) \\
\frac{d}{dx} \text{sech}(x) &= -\text{sech}(x) \tanh(x) \\
\frac{d}{dx} \text{csch}(x) &= -\text{csch}(x) \coth(x)
\end{align*}
\]

We can also define inverse hyperbolic functions by

\[
\begin{align*}
\sinh^{-1}(\sinh(x)) &= x = \sinh(\sinh^{-1}(x)) \\
\cosh^{-1}(\cosh(x)) &= x = \cosh(\cosh^{-1}(x)) \\
\tanh^{-1}(\tanh(x)) &= x = \tanh(\tanh^{-1}(x))
\end{align*}
\]
These turn out to have fairly simply formulas that we can state directly in terms of $x$:

\[
\begin{align*}
\sinh^{-1}(x) &= \ln \left(x + \sqrt{x^2 + 1}\right) \\
\cosh^{-1}(x) &= \ln \left(x + \sqrt{x^2 - 1}\right) \\
\tanh^{-1}(x) &= \frac{1}{2} \ln \left(\frac{1 + x}{1 - x}\right), \quad x^2 < 1 \\
\coth^{-1}(x) &= \frac{1}{2} \ln \left(\frac{1 - x}{1 + x}\right), \quad x^2 > 1 \\
\sech^{-1}(x) &= \log \left(\frac{1 + \text{sgn}(x) \sqrt{1 + x^2}}{x}\right) \\
\csch^{-1}(x) &= \log \left(\frac{1 + \text{sgn}(x) \sqrt{1 - x^2}}{x}\right)
\end{align*}
\]

The derivatives of the inverse hyperbolic functions are given by the following table

\[
\begin{align*}
\frac{d}{dx} \sinh^{-1}(x) &= \frac{1}{\sqrt{1 + x^2}} \\
\frac{d}{dx} \cosh^{-1}(x) &= \frac{1}{\sqrt{x^2 - 1}} \\
\frac{d}{dx} \tanh^{-1}(x) &= \frac{1}{1 - x^2} \\
\frac{d}{dx} \coth^{-1}(x) &= \frac{1}{1 - x^2} \\
\frac{d}{dx} \sech^{-1}(x) &= -\frac{1}{|x| \sqrt{x^2 + 1}} \\
\frac{d}{dx} \csch^{-1}(x) &= -\frac{1}{|x| \sqrt{1 - x^2}}
\end{align*}
\]