Part II. Team Round

1. For an integer $B \geq 2$, the **base B representation** of a number n is a string of "base B digits" (integers between zero and $(B-1)$) $d_\ell d_{\ell-1} \ldots d_2 d_1 d_0$ with the property that $n = d_0 \cdot B^0 + d_1 \cdot B^1 + d_2 \cdot B^2 + \cdots + d_\ell \cdot B^\ell$.

   For example, the number 25 has base ten representation 25 because $25 = 2 \cdot 10^1 + 5 \cdot 10^0$, base two representation 11001 because $25 = 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$, and base three representation 221, because $25 = 2 \cdot 3^2 + 2 \cdot 3^1 + 1 \cdot 3^0$.

   A. Find the base two, base three, base eight, and base sixteen representations of the number whose base ten representation is 2014. (In base sixteen, the "digits" are $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, d, e, f$.)

   B. Prove that, if $B \geq 2$, every positive integer has exactly one base B representation.

Solution: To find the base two representation of 2014, we repeatedly divide by 2:

$$\begin{aligned}
2014 &= 2(1007) + 0 \\
1007 &= 2(503) + 1 \\
503 &= 2(251) + 1 \\
251 &= 2(125) + 1 \\
125 &= 2(62) + 1 \\
62 &= 2(31) + 0 \\
31 &= 2(15) + 1 \\
15 &= 2(7) + 1 \\
7 &= 2(3) + 1 \\
3 &= 2(1) + 1 \\
1 &= 2(0) + 1.
\end{aligned}$$

Plugging each right-hand side into the next left-hand side and expanding, we eventually get

$$2014 = 2^{10} * 1 + 2^9 * 1 + 2^8 * 1 + 2^7 * 1 + 2^6 * 1 + 2^5 * 0 + 2^4 * 1 + 2^3 * 1 + 2^2 * 1 + 2^1 * 1 + 0.$$

The base two representation is obtained by writing the remainders in reverse order of their appearance: $\boxed{11111011110}$.

For base three, we repeatedly divide by 3:

$$\begin{aligned}2014 &= 3(671) + 1\\ 671 &= 3(223) + 2\\ 223 &= 3(74) + 1\\ 74 &= 3(24) + 2\\ 24 &= 3(8) + 0\\ 8 &= 3(2) + 2\\ 2 &= 3(0) + 2.\end{aligned}$$

Thus the base three representation of 2014 is $\boxed{2202121}$.

For base eight, we repeatedly divide by 8:

$$\begin{aligned}2014 &= 8(251) + 6\\ 251 &= 8(31) + 3\\ 31 &= 8(3) + 7\\ 3 &= 8(0) + 3\end{aligned}$$

Thus the base eight representation of 2014 is $\boxed{3736}$.

For base sixteen, we repeatedly divide by 16:

$$\begin{aligned}2014 &= 16(125) + 14\\ 125 &= 16(7) + 13\\ 7 &= 16(0) + 7.\end{aligned}$$

Thus the base sixteen representation of 2014 is 7(13)(14), i.e., $\boxed{7de}$.

Note that the base eight and sixteen representations can be obtained directly from the base two representation: $8 = 2^3$, so we group the base two digits by threes to get $11111011110 = (011)(111)(011)(110) = 3736$. Similarly $16 = 2^4$, so we group the base two digits by fours: $11111011110 = (0111)(1101)(1110) = 7de$.

To prove that every number has a base B representation, observe that our repeated-division algorithm above works for every positive integer n and every $B \geq 2$.

To prove that the base B representation is unique, suppose that some integer has two different base B representations. Then there is a smallest such integer; call it n. We have:

$$n = d_0 + d_1 B + d_2 B^2 + \cdots + d_r B^r$$
$$n = c_0 + c_1 B + c_2 B^2 + \cdots + c_s B^s.$$

First, observe that if $r < s$, we may set $d_{r+1} = d_{r+2} = \cdots = d_s = 0$ (and likewise if $s < r$), so we can assume $r = s$. Second, if $c_r \neq d_r$, we could subtract $d_r B^r$ from both expressions to obtain two different base B representations of the smaller number $n - d_r B^r$. Thus, we may assume that $c_r \neq d_r$.

Now, if we subtract the two representations from one another, we obtain

$$0 = (d_0 - c_0) + (d_1 - c_1)B + (d_2 - c_2)B^2 + \cdots + (d_r - c_r)B^r.$$

Set $f_i = d_i - c_i$, and observe that $-B < f_i < B$ for all i and $f_r \neq 0$. Then, rearranging, we have

$$-f_r B^r = f_0 + f_1 B + f_2 B^2 + \cdots + f_{r-1} B^{r-1}.$$

Taking the absolute value of everything in sight yields

$$|f_r| B^r \leq |f_0| + |f_1| B + |f_2| B^2 \cdots + |f_{r-1}| B^{r-1}.$$

Thus we have

$$\begin{aligned}
B^r &\leq |f_r| B^r \\
&\leq |f_0| + |f_1| B + |f_2| B^2 \cdots + |f_{r-1}| B^{r-1} \\
&\leq (B-1) + (B-1)B + (B-1)B^2 + \cdots + (B-1)B^{r-1} \\
&\leq (B-1)(1 + B + B^2 + \ldots B^{r-1}) \\
&= B^r - 1.
\end{aligned}$$

But clearly $B^r \not\leq B^r - 1$, so our assumption that some number has two different base B representations must have been wrong.

2.

A. Partition the set $\{1, 2\}$ into two disjoint subsets $S = \{s_1\}$ and $T = \{t_1\}$ with the property $s_1^0 = t_1^0$.

B. Partition the set $\{1, 2, 3, 4\}$ into two disjoint subsets $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$ with the properties

$$\begin{aligned} s_1^0 + s_2^0 &= t_1^0 + t_2^0 \\ s_1^1 + s_2^1 &= t_1^1 + t_2^1. \end{aligned}$$

C. Partition the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ into two disjoint subsets $S = \{s_1, s_2, s_3, s_4\}$ and $T = \{t_1, t_2, t_3, t_4\}$ with the properties

$$\begin{aligned} s_1^0 + s_2^0 + s_3^0 + s_4^0 &= t_1^0 + t_2^0 + t_3^0 + t_4^0 \\ s_1^1 + s_2^1 + s_3^1 + s_4^1 &= t_1^1 + t_2^1 + t_3^1 + t_4^1 \\ s_1^2 + s_2^2 + s_3^2 + s_4^2 &= t_1^2 + t_2^2 + t_3^2 + t_4^2. \end{aligned}$$

D. Partition the set $\{1, \ldots, 32\}$ into two disjoint subsets $S = \{s_1, \ldots, s_{16}\}$ and $T = \{t_1, \ldots t_{16}\}$ with the properties

$$\begin{aligned} s_1^0 + \cdots + s_{16}^0 &= t_1^0 + \cdots + t_{16}^0 \\ s_1^1 + \cdots + s_{16}^1 &= t_1^1 + \cdots + t_{16}^1 \\ s_1^2 + \cdots + s_{16}^2 &= t_1^2 + \cdots + t_{16}^2 \\ s_1^3 + \cdots + s_{16}^3 &= t_1^3 + \cdots + t_{16}^3 \\ s_1^4 + \cdots + s_{16}^4 &= t_1^4 + \cdots + t_{16}^4. \end{aligned}$$

(Just state your answer. You don't need to prove it.)

Solution: For part (a), the solution is $S = \{1\}$, $T = \{2\}$, which we verify by checking $1^0 = 1$ and $2^0 = 1$.

For part (b), the solution is $S = \{1, 4\}$, $T = \{2, 3\}$. Checking, we have $1^0 + 4^0 = 2^0 + 3^0 = 2$, and $1^1 + 4^1 = 2^1 + 3^1 = 5$.

For part (c), the solution is $S = \{1, 4, 6, 7\}$, $T = \{2, 3, 5, 8\}$. Checking, the zeroeth powers add to 4, the first powers add to 18, and the squares add to 102.

Before attacking part (d), we need to find a pattern. One thing we might notice is that the solutions to parts (b) and (c) contain the solutions to parts (a) and (b) as subsets. Another important observation is that we obtained the solution to (b) by placing 3 and 4 in S and T in the opposite order that 1 and 2 were placed, and then we obtained the solution to (c) by placing 5 through 8 in the opposite order that we had placed 1 through 4. That is:

S	T	1	2
T	S	3	4

S	T	T	S	1	2	3	4
T	S	S	T	5	6	7	8

We might try to continue this pattern to partition the numbers up to 16. If we do, we get

S	T	T	S	T	S	S	T	1	2	3	4	5	6	7	8
T	S	S	T	S	T	T	S	9	10	11	12	13	14	15	16

Indeed, if we set $S = \{1, 4, 6, 7, 10, 11, 13, 16\}$ and $T = \{2, 3, 5, 8, 9, 12, 14, 15\}$, then the zeroeth powers of each set add to 8, the first powers add to 68, the squares add to 748, and the cubes add to 9248.

Continuing the pattern, we intuit that the answer to (d) is

$$S = \{1, 4, 6, 7, 10, 11, 13, 16, 18, 19, 21, 24, 25, 28, 30, 31\},$$
$$T = \{2, 3, 5, 8, 9, 12, 14, 15, 17, 20, 22, 23, 26, 27, 29, 32\}.$$

In fact, the zeroeth powers here add to 16, the first powers add to 264, the squares add to 8536, the cubes add to 139392, and the fourth powers add to 3623048.

These sets S and T are called the **Thue-Morse sequence**, and the pattern continues: We can partition the first 64 numbers so that the zeroeth, first, second, third, fourth, and fifth powers of the elements in each set add to the same thing, and so on.

3. Say that a subset of $\{1, 2, \ldots, n\}$ is **clean** if it does not contain both a number and its double. For example, $\{1, 3, 5, 7, 8\}$ is clean because it does not contain 2, 6, 10, 14, or 16, but $\{1, 2, 3\}$ is not clean because it contains both 1 and 2.

   For a positive integer n, let $C(n)$ be the largest possible size of a clean subset of $\{1, 2, \ldots, n\}$. Thus $C(1) = 1$ ($\{1\}$ is clean), $C(2) = 1$ ($\{1\}$ and $\{2\}$ are clean, but $\{1, 2\}$ isn't), and $C(3) = 2$ ($\{1, 3\}$ is clean).

   A. Find $C(4)$, $C(6)$, and $C(11)$.

   B. Find $C(2014)$.

Solution: For an odd number k, let $A_k = \{k, 2k, 4k, 8k, \ldots\}$ be the powers of two times k. Then a set is clean if and only if it doesn't contain consecutive elements of any A_k. The easiest way to do this while being as large as possible is to contain $k, 4k, 16k, 64k, \ldots$ but not $2k, 8k, 32k, \ldots$ for every odd number. Thus, the largest clean subset of $\{1, 2, \ldots, n\}$ is the one consisting of all odd numbers less than or equal to n, all numbers divisible by 4 but not 8 and less than or equal to n, all numbers divisible by 16 but not 32 and less than or equal to n, and so on.

For $n = 4$, this is $\{1, 3, 4\}$, so $\boxed{C(4) = 3}$.

For $n = 6$, our set is $\{1, 3, 4, 5\}$, so $\boxed{C(6) = 4}$.

For $n = 11$, our set $\{1, 3, 4, 5, 7, 9, 11\}$, so $\boxed{C(11) = 7}$.

For $n = 2014$, we don't want to write out the set explicitly. But we can describe it as follows:

Start with all the numbers from 1 to 2014. Then remove the even numbers. Then add back in the multiples of 4. Then remove the multiples of 8. Then add back in the multiples of 16. Continue until the set stops changing.

Thus the size of our largest clean set is $2014 - \lfloor \frac{2014}{2} \rfloor + \lfloor \frac{2014}{4} \rfloor - \ldots$.

Piggybacking on our work from problem 1, we have

$$C(2014) = 2014 - 1007 + 503 - 251 + 125 - 62 + 31 - 15 + 7 - 3 + 1 = \boxed{1343}.$$

4. Let (a, b) be a pair of real numbers such that the equations $x^2 - ax + b = 0$ and $x^2 - bx + a = 0$ each have two positive integers as roots.

   A. Prove that a and b are both positive integers.

   B. Find a number C with the property that $|a - b| < C$.

   C. Find all possible pairs (a, b).

Solution: Let r_1 and r_2 be the roots of $x^2 - ax + b$, and s_1 and s_2 be the roots of $x^2 - bx + a$. By assumption, all four roots are positive integers.

For (a), observe that $b = r_1 r_2$ and $a = s_1 s_2$ are each products of positive integers. Alternatively, $a = r_1 + r_2$ and $b = s_1 + s_2$ are both sums of positive integers.

For (b), suppose $a \geq b$. Then $r_1 + r_2 \geq r_1 r_2$. Since r_1 and r_2 are positive integers, it follows that either $r_1 = r_2 = 2$ (meaning $a = b = 4$) or $a = b + 1$.

(If instead $a \leq b$, we similarly get $a = b = 4$ or $b = a + 1$.)

Consequently, one correct choice for C is $\boxed{2}$.

For (c), there are three possibilities: $a = b = 4$, $a = b + 1$, and $a = b - 1$.

If $a = b = 4$, then both equations have a double root at $x = 2$, which is a positive integer.

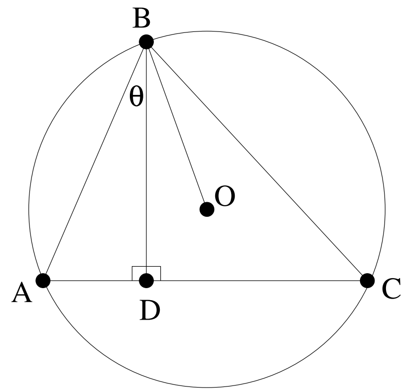
If $a = b+1$, then the discriminant $b^2 - 4a = b^2 - 4b - 4 = (b-2)^2 - 8$ is a perfect square. The only perfect squares that differ by 8 are 1 and 9, so we conclude that $(b - 2) = 3$ and $b = 5$. In this case $x^2 - ax + b = x^2 - 6x + 5$ and $x^2 - bx + a = x^2 - 5x + 6$ both do have positive integer roots.

If $a = b - 1$, we similarly conclude that $a = 5$ and $b = 6$.

Thus, the solutions are $\boxed{(2, 2),\ (5, 6),\text{ and }(6, 5)}$.

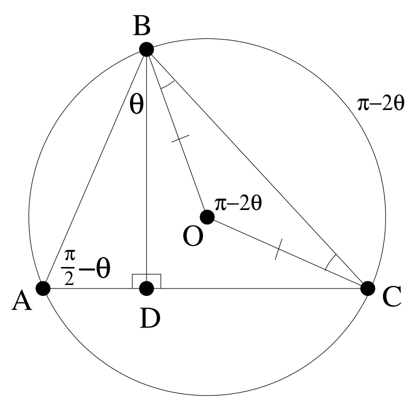
There are other methods once we've determined that r_1 and r_2 differ by at most one. For example, we can show that the sum and product differ by one only in the cases $(2, 3)$ and $(1, n)$, then analyze the other quadratic to eliminate most of the $(1, n)$ cases.

5. In the figure below, triangle ABC is inscribed in circle O, and BD is the altitude from B to AC. If angle ABD has measure θ, find the measure of angle OBC.



Solution: Observe that angle BAD has measure $\frac{\pi}{2} - \theta$, and that it cuts off arc BC on the far side of the circle. Thus arc BC measures $\pi - 2\theta$.

Now draw OC.



Angle BOC also cuts off arc BC, but is at the center of the circle, so it measures $\pi - 2\theta$ as well. Now triangle BOC is isosceles (since OB and OC are both radii), so angles OBC and OCB are equal.

Thus $OBC = \frac{1}{2}(\pi - BOC) = \frac{1}{2}(2\theta) = \boxed{\theta}$.