Part II. Team Round

1. For an integer $B \geq 2$, the **base** B **representation** of a number n is a string of "base B digits" (integers between zero and $(B-1)$) $d_\ell d_{\ell-1} \ldots d_2 d_1 d_0$ with the property that $n = d_0 \cdot B^0 + d_1 \cdot B^1 + d_2 \cdot B^2 + \cdots + d_\ell \cdot B^\ell$.

   For example, the number 25 has base ten representation 25 because $25 = 2 \cdot 10^1 + 5 \cdot 10^0$, base two representation 11001 because $25 = 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$, and base three representation 221, because $25 = 2 \cdot 3^2 + 2 \cdot 3^1 + 1 \cdot 3^0$.

   A. Find the base two, base three, base eight, and base sixteen representations of the number whose base ten representation is 2014. (In base sixteen, the "digits" are $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, d, e, f$.)

   B. Prove that, if $B \geq 2$, every positive integer has exactly one base B representation.

2.

A. Partition the set $\{1, 2\}$ into two disjoint subsets $S = \{s_1\}$ and $T = \{t_1\}$ with the property $s_1^0 = t_1^0$.

B. Partition the set $\{1, 2, 3, 4\}$ into two disjoint subsets $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$ with the properties

$$\begin{aligned} s_1^0 + s_2^0 &= t_1^0 + t_2^0 \\ s_1^1 + s_2^1 &= t_1^1 + t_2^1. \end{aligned}$$

C. Partition the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ into two disjoint subsets $S = \{s_1, s_2, s_3, s_4\}$ and $T = \{t_1, t_2, t_3, t_4\}$ with the properties

$$\begin{aligned} s_1^0 + s_2^0 + s_3^0 + s_4^0 &= t_1^0 + t_2^0 + t_3^0 + t_4^0 \\ s_1^1 + s_2^1 + s_3^1 + s_4^1 &= t_1^1 + t_2^1 + t_3^1 + t_4^1 \\ s_1^2 + s_2^2 + s_3^2 + s_4^2 &= t_1^2 + t_2^2 + t_3^2 + t_4^2. \end{aligned}$$

D. Partition the set $\{1, \ldots, 32\}$ into two disjoint subsets $S = \{s_1, \ldots, s_{16}\}$ and $T = \{t_1, \ldots t_{16}\}$ with the properties

$$\begin{aligned} s_1^0 + \cdots + s_{16}^0 &= t_1^0 + \cdots + t_{16}^0 \\ s_1^1 + \cdots + s_{16}^1 &= t_1^1 + \cdots + t_{16}^1 \\ s_1^2 + \cdots + s_{16}^2 &= t_1^2 + \cdots + t_{16}^2 \\ s_1^3 + \cdots + s_{16}^3 &= t_1^3 + \cdots + t_{16}^3 \\ s_1^4 + \cdots + s_{16}^4 &= t_1^4 + \cdots + t_{16}^4. \end{aligned}$$

(Just state your answer. You don't need to prove it.)

3. Say that a subset of $\{1, 2, \ldots, n\}$ is **clean** if it does not contain both a number and its double. For example, $\{1, 3, 5, 7, 8\}$ is clean because it does not contain 2, 6, 10, 14, or 16, but $\{1, 2, 3\}$ is not clean because it contains both 1 and 2.

   For a positive integer n, let $C(n)$ be the largest possible size of a clean subset of $\{1, 2, \ldots, n\}$. Thus $C(1) = 1$ ($\{1\}$ is clean), $C(2) = 1$ ($\{1\}$ and $\{2\}$ are clean, but $\{1, 2\}$ isn't), and $C(3) = 2$ ($\{1, 3\}$ is clean).

   A. Find $C(4)$, $C(6)$, and $C(11)$.

   B. Find $C(2014)$.

4. Let (a, b) be a pair of real numbers such that the equations $x^2 - ax + b = 0$ and $x^2 - bx + a = 0$ each have two positive integers as roots.

   A. Prove that a and b are both positive integers.

   B. Find a number C with the property that $|a - b| < C$.

   C. Find all possible pairs (a, b).

5. In the figure below, triangle ABC is inscribed in circle O, and BD is the altitude from B to AC. If angle ABD has measure θ, find the measure of angle OBC.