Comprehensive Exam, Geometric and Algebraic Topology, August 2007

Attempt all of the problems. They have equal weight. 60% guarantees passing.

Unless otherwise indicated all homology groups are with integer coefficients.

1. Let M be the surface obtained by identifying the edges of an octagon using the pattern $abcbdc^{-1}da^{-1}$. Compute the Euler characteristic of M, then identify which surface this is.

2. Write down presentations for $\pi_1(X)$. In parts (c) and (d) also say in words what the group is.

   (a) $X = T$, where T is the torus.

   (b) $X = T\#T$, where $\#$ denotes connected sum.

   (c) $X = T \times T$, where $\times$ denotes Cartesian product.

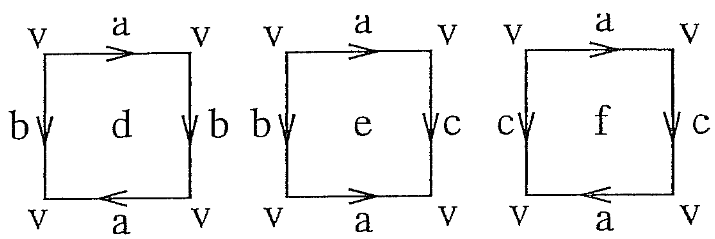
   (d) $X = T \vee T$, where $\vee$ denotes the union along a single point.

3. Let $p : \widetilde{X} \to X$ be a covering space map, where X and $\widetilde{X}$ are path connected and locally path connected. Let $x_0, x_1 \in X$. Prove that there is a one to one correspondence $\varphi : p^{-1}(x_0) \to p^{-1}(x_1)$.

4. Let $G = \mathbb{Z}_2 * \mathbb{Z}_2$, where the generators for the two factors are a and b. Prove that if $x \in G$, $x \neq 1$, and $x^n = 1$ for some $n > 1$, then x is conjugate to a or b.

5. Let $X = X_1 \cup X_2$, where X_1 and X_2 are tori and $A = X_1 \cap X_2$ is a simple closed curve which does not bound a disk in X_1 but does bound a disk in X_2. Use the Mayer-Vietoris sequence to compute $H_1(X)$ and $H_2(X)$.

6. Let X be the CW-complex indicated below. It has one 0-cell v, three 1-cells a, b, and c, and three 2-cells d, e, and f. Use cellular homology to compute $H_1(X)$, and $H_2(X)$.



7. Let X be a path connected space with $H_1(X) = \mathbb{Z}_6$ and $H_2(X) = \mathbb{Z} \oplus \mathbb{Z}_{15}$.

   (a) Compute $H_1(X;\mathbb{Z}_3)$ and $H_2(X;\mathbb{Z}_3)$.

   (b) Compute $H^1(X)$ and $H^2(X)$.

8. Suppose the space X in Problem 7 is a closed, orientable 4-manifold. Compute $H_3(X)$, $H_4(X)$, $H^3(X)$, and $H^4(X)$.