DOCTORAL EXAM IN TOPOLOGY: GEOMETRIC TOPOLOGY

You should attempt all the problems. Partial credit will be given for serious efforts.

1. Let $p : \widetilde{X} \to X$ be a covering map. Let A be a connected topological space. Suppose f and g are continuous functions from A to $\widetilde{X}$ such that $p \circ f = p \circ g$. Prove that if there is a point $a_0 \in A$ such that $f(a_0) = g(a_0)$, then $f(a) = g(a)$ for all $a \in A$.

2. Consider the following subsets of $\mathbf{R}^3$:

   $Z = \{(x, y, z) \mid x = 0, y = 0\}$.

   $V = \{(x, y, z) \mid x = 0, y = 3\}$.

   $C_1 = \{(x, y, z) \mid x^2 + y^2 = 1, z = 0\}$.

   $C_2 = \{(x, y, z) \mid x^2 + y^2 = 4, z = 0\}$.

   Find the fundamental groups of the following spaces. (In all cases these are familiar groups; you should say what the group is, not just give a presentation.)

   (a) $\mathbf{R}^3 - Z$
   (b) $\mathbf{R}^3 - (Z \cup V)$
   (c) $\mathbf{R}^3 - (Z \cup C_1)$
   (d) $\mathbf{R}^3 - (Z \cup C_1 \cup C_2)$
   (e) $\mathbf{R}^3 - (Z \cup V \cup C_1)$

3. Draw pictures of all the compact, connected, orientable surfaces S (possibly with boundary) having $\chi(S) = -4$.

4. Let $X = S^1 \vee S^1$ be the union of two copies of S^1 joined at a single point.

   (a) Draw pictures of all the 2-sheeted coverings of X (up to equivalence). Give a reason why your list is complete. .
   (b) Use (a) to determine the rank of a subgroup of index two of a free group of rank 2.

DOCTORAL EXAM IN TOPOLOGY: ALGEBRAIC TOPOLOGY

You should attempt all the problems. Partial credit will be given for serious efforts.

1. Let X be the space obtained by attaching two 2-cells to the figure 8 space $S^1 \vee S^1$ by the maps given by the words ab and $a(ba)^2a^{-1}$. Compute all the groups $H_p(X)$.

2. Let X and Y be copies of the real projective plane P^2. Suppose $X \cap Y$ is a simple closed curve which is essential in both X and Y. Compute all the groups $H_p(X \cup Y)$.

3. Let T be a torus and C an essential simple closed curve in T. Compute all the groups $H_p(T, C)$.

4. Prove that the unit n-sphere S^n is not a retract of the closed unit ball B^{n+1}. Then use this fact to prove the Brouwer Fixed Point Theorem.

Aug. 03

DOCTORAL EXAM ON TOPOLOGY

Geometric Topology and Algebraic Topology

You should attempt all the problems. Partial credit will be give for serious efforts

(1) Let $X = S^1 \vee P^2$ be the wedge sum of a circle and a projective plane. List all the 3–fold covering spaces of X.

(2) Compute the fundamental group of the figure eight and draw a piece of its universal covering space.

(3) Let X be the union of the following three (3) sets from R^3: the 2-sphere $\{(x,y,z) \in R^3 : x^2+y^2+z^2 = 1\}$; the horizontal disk $\{(x,y,z) \in R^3, x^2+y^2 \leq 1, z = 0\}$; and the vertical line segment in the z-axis $\{(0,0,z) \in R^3, -1 < z < 1\}$. Compute the fundamental group $\pi_1(X)$ and integral homology $H_*(X;Z)$ of the space X.

(4) Let M be a closed, orientable 3–manifold.

a) Prove that $H_2(M;Z)$ is a free abelian group;

b) Give an exmaple of orientable closed 3–manifold with torsion elements in $H_1(M;Z)$.

(5) Let $\{C_n, \partial_n\}$ and $\{C'_n, \partial'_n\}$ be chain complexes. Suppose $\{\phi_n : C_n \to C'_n\}$ and $\{\psi_n : C_n \to C'_n\}$ are chain maps. Let $\{\Delta_n : C_n \to C'_{n+1}\}$ be a chain homotopy between ϕ_n and ψ_n:

$$\partial'_{n+1} \circ \Delta_n + \Delta_{n-1} \circ \partial_n = \phi_n - \psi_n.$$

Prove that in this case the two chain maps induce the same homomorphism from the homology of $\{C_n, \partial_n\}$ to that of $\{C'_n, \partial'_n\}$.

(6) Let M be an orientable 3–manifold with boundary ∂M. Prove that the kernel of the map $i_* : H_1(\partial M;Z) \to H_1(M;Z)$ is not trivial, where $i : \partial M \to M$ is the inclusion.