Aug. 03

DOCTORAL EXAM ON TOPOLOGY

Geometric Topology and Algebraic Topology

You should attempt all the problems. Partial credit will be give for serious efforts

(1) Let $X = S^1 \vee P^2$ be the wedge sum of a circle and a projective plane. List all the 3–fold covering spaces of X.

(2) Compute the fundamental group of the figure eight and draw a piece of its universal covering space.

(3) Let X be the union of the following three (3) sets from R^3: the 2-sphere $\{(x,y,z) \in R^3 : x^2+y^2+z^2 = 1\}$; the horizontal disk $\{(x,y,z) \in R^3, x^2+y^2 \leq 1, z = 0\}$; and the vertical line segment in the z-axis $\{(0,0,z) \in R^3, -1 < z < 1\}$. Compute the fundamental group $\pi_1(X)$ and integral homology $H_*(X;Z)$ of the space X.

(4) Let M be a closed, orientable 3–manifold.

a) Prove that $H_2(M;Z)$ is a free abelian group;

b) Give an exmaple of orientable closed 3–manifold with torsion elements in $H_1(M;Z)$.

(5) Let $\{C_n, \partial_n\}$ and $\{C'_n, \partial'_n\}$ be chain complexes. Suppose $\{\phi_n : C_n \to C'_n\}$ and $\{\psi_n : C_n \to C'_n\}$ are chain maps. Let $\{\Delta_n : C_n \to C'_{n+1}\}$ be a chain homotopy between ϕ_n and ψ_n:

$$\partial'_{n+1} \circ \Delta_n + \Delta_{n-1} \circ \partial_n = \phi_n - \psi_n.$$

Prove that in this case the two chain maps induce the same homomorphism from the homology of $\{C_n, \partial_n\}$ to that of $\{C'_n, \partial'_n\}$.

(6) Let M be an orientable 3–manifold with boundary ∂M. Prove that the kernel of the map $i_* : H_1(\partial M; Z) \to H_1(M;Z)$ is not trivial, where $i : \partial M \to M$ is the inclusion.