DOCTORAL EXAM - TOPOLOGY - JANUARY 2003

You should attempt all the problems. Partial credit will be given for serious efforts.

1. Let $\mathcal{T}$ be a triangulation of S^2 having n vertices of order 3 and k vertices of order 4, where $n \geq 0$ and $k \geq 0$. List all the possibilities for the pair (n, k). Prove your answer.

2. Let T_1 and T_2 be tori (copies of $S^1 \times S^1$). Let C_i be an essential simple closed curve on T_i. (You may assume that it has the form $x_i \times S^1$ in T_i.) Let X be the space obtained from T_1 and T_2 by identifying C_1 with C_2. (So after the identification T_1 and T_2 intersect in a circle C.)

   (a) Compute $\pi_1(X)$. Give a presentation and also describe in words what the group is.

   (b) Compute $H_1(X)$ and $H_2(X)$.

   Hint: You can either view X as a union of two spaces as described above and apply the appropriate theorems or you can take a shortcut by viewing X as being constructed in a different way and apply another set of theorems. (Draw a picture and stare at it!)

3. Let X be a path connected space with $\pi_1(X)$ finite. Prove that any map $f : X \to S^1$ is homotopic to a constant map. (Hint: universal covering space.)

4. Let X be a path connected space with $H_1(X) = \mathbf{Z}_2$ and $H_2(X) = \mathbf{Z}$. (All homology and cohomology groups in this problem have integer coefficients.)

   (a) Compute $H^1(X)$ and $H^2(X)$.

   (b) Now assume that X is a closed, orientable 5-manifold. Compute $H_3(X)$, $H_4(X)$, $H^3(X)$, and $H^4(X)$.

5. (a) Let X be the figure eight space (two copies of S^1 joined at a single point). Give an example of a connected 3-sheeted covering space $\widetilde{X}$ of X.

   (b) Let M be a closed, orientable surface of genus two. Give an example of a connected 3-sheeted covering space $\widetilde{M}$ of M. What is the genus of $\widetilde{M}$?

   (c) Prove that for any connected 3-sheeted covering space $\widetilde{M}$ of M the genus of $\widetilde{M}$ is the same as the genus of your example. (Hint: Euler characteristic.)

6. Let $f, g : X \to X$ be maps such that for some point $x_0 \in X$ we have $f(x_0) = g(x_0) = x_0$. Suppose $H : X \times [0,1] \to X$ is a homotopy such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. It is NOT assumed that $H(x_0, t) = x_0$ for all $t \in [0, 1]$. Prove that for each $\alpha \in \pi_1(X, x_0)$ we have that $f_*(\alpha)$ and $g_*(\alpha)$ are conjugate in $\pi_1(X, x_0)$.