DOCTORAL EXAM - TOPOLOGY - AUGUST 2002

1. Prove that the sphere S^{n-1} is not a retract of the closed ball B^n. Then state and prove the Brouwer Fixed Point Theorem.

2. List, up to isomorphism, all the connected 2-sheeted covering spaces of the figure eight space. What is the fundamental group of each covering space? Generalize this observation to find the rank of a subgroup H of index k of a free group G of rank n.

3. Let $k \geq 2$. Recall that a k-fold crosscap X_k is the space obtained by attaching a closed disk D to a circle S^1 by the map $f : \partial D \to S^1$ given by $f(z) = z^k$. (Here D is the unit disk and S^1 the unit circle in the complex plane.) For example, X_2 is the projective plane. Compute $H_p(X_k)$, $H_p(X_k, S^1)$, $H_p(X_k; \mathbf{Z}_k)$, and $H_p(X_k, S^1; \mathbf{Z}_k)$ for all $p \geq 0$.

4. The suspension ΣX of a space X is the space obtained from $X \times [-1, 1]$ by identifying $X \times \{-1\}$ to a point (the "south pole") and $X \times \{1\}$ to a point (the "north pole"). Thus ΣX is just the union of two copies of the cone CX on X along a common copy of X. For example, when $X = S^1$, the cone CS^1 is homeomorphic to a disk; then ΣS^1 is the union of two disks along their boundaries and so is homeomorphic to S^2. How are the homology groups of X and ΣX related? Prove your answer. Hint: Recall that a cone is contractible.

5. Let x and y be points in a space X. Suppose g is a path in X from x to y. Let $\gamma = [g]$ be the path homotopy class of g. There is a change of basepoint isomorphism $\widehat{\gamma} : \pi_1(X, x) \to \pi_1(X, y)$. How is $\widehat{\gamma}$ defined? That is, given an element α of $\pi_1(X, x)$, write down the formula for $\widehat{\gamma}(\alpha)$ in terms of γ and α. Then prove that $\widehat{\gamma}$ is an isomorphism.

6. For $g \geq 0$, $b \geq 0$ let $M_{g,b}$ be the compact, connected, orientable surface of genus g with b boundary components. That is, $M_{g,b}$ is the surface obtained by removing the interiors of b disjoint closed disks from the connected sum of g tori. For $h \geq 1$, $b \geq 0$ let $N_{h,b}$ be the compact, connected, non-orientable surface obtained by removing the interiors of b disjoint closed disks from the connected sum of h projective planes. Now suppose that S is a compact, connected surface with $\chi(S) = -2$. Using the notation given above list all the possibilities for S. For each

of these surfaces describe the fundamental group as follows. If it is free, give its rank. If it is not free, give a presentation.

7. Give examples of spaces X and Y such that:

   (a) $\pi_1(X) = \pi_1(Y)$, but $H_p(X) \neq H_p(Y)$ for some p.

   (b) $\pi_1(X) \neq \pi_1(Y)$, but $H_p(X) = H_p(Y)$ for all p.

   (c) $\pi_1(X) = \pi_1(Y)$, and $H_p(X) = H_p(Y)$ for all p, but X and Y are not homeomorphic.

   (d) $\pi_1(X) = \pi_1(Y)$, and $H_p(X) = H_p(Y)$ for all p, but X and Y are not homotopy equivalent.

8. Let X and Y be compact, connected, orientable surfaces without boundary. Suppose $f : X \to Y$ is a map such that the image of the induced homomorphism $f_* : \pi_1(X) \to \pi_1(Y)$ has infinite index in $\pi_1(Y)$. Prove that the induced map $f_\# : H_2(X) \to H_2(Y)$ is the zero map. Hint: Use a covering space of Y.