DOCTORAL EXAM ON TOPOLOGY, AUGUST 2001

You should attempt all the problems. Partial credit will be give for serious efforts

1. What surfaces are represented by $2n$-gons with the edges identified in pairs according to the symbols

   (a) $w_1 = a_1a_2a_3a_4a_5a_1^{-1}a_2^{-1}a_3^{-1}a_4^{-1}a_5^{-1}$. Find the surface S_1 for this word.

   (b) $w_2 = a_1a_2a_3a_4a_1^{-1}a_2^{-1}a_3^{-1}a_4^{-1}$. Find the surface S_2 for this word.

2. Let S be a closed surface with $H_1(S;Z) = Z \oplus Z \oplus Z \oplus Z_2$, where Z is the integers. Find the fundamental group of S and the Euler characteristic of S.

3. Construct two inequivalent 3-fold covering spaces of the figure eight $(S^1 \vee S^1)$. Prove your 3-fold covering spaces are not equivalent as covering spaces. Are they equivalent as topological spaces ?

4. Suppose X is path-connected, locally path-connected and $\pi_1(X)$ consists only of torsion elements. Prove that every continuous map $X \to T^n = S^1 \times S^1 \times \cdots S^1$ is nullhomotopic.

5. Let X be obtained from the figure eight by attaching two 2-cells by the words $a^{-1}b^{-1}$ and $b(ab)^{-2}b^{-1}$. Compute $H_i(X;Z)$ for all $i \geq 0$.

6. Compute $H_i(X_1 \times X_2; Z)$, where X_1 is the closed oriented surface of genus 2, and X_2 is the Klein bottle.

7. Let Z denote the group of integers and suppose G an arbitrary Abelian group.

   (a) State the universal coefficient theorem, which gives a formula for the cohomology group $H^p(X;G)$ in terms of the homology groups $H_i(X;Z)$ and the group G.

(b) Now assume that X is a compact, connected, orientable n-manifold without boundary. State the Poincare duality theorem relating the homology and cohomology groups of X; use coefficient group Z.

(c) Use (a) and (b) to show that $H_{n-1}(X; Z)$ is a free Abelian group.