DOCTORAL EXAM - TOPOLOGY - JANUARY 2000

1. Suppose X and Y are path connected spaces. Consider the following statements:

   (a) X and Y are homeomorphic.

   (b) X and Y are homotopy equivalent.

   (c) $\pi_1(X)$ and $\pi_1(Y)$ are isomorphic.

   How are these statements related? That is, which implies which? For each implication which does not hold, give a counterexample to the implication.

2. Let $p : \widetilde{M} \to M$ be a k-sheeted covering map, where $\widetilde{M}$ and M are closed, connected, orientable surfaces and M has genus g. What is the genus $\tilde{g}$ of $\widetilde{M}$? What is $\tilde{g}$ when $k = 3$ and $g = 2$?

3. Regard S^1 as the set of complex numbers z with $|z| = 1$. Let X, $\widetilde{X}$, and Y be copies of S^1. Let $p : \widetilde{X} \to X$ be the covering map given by $p(z) = z^2$. For each of the following maps $f : Y \to X$, determine whether or not there is a lifting $g : Y \to \widetilde{X}$. If there is a lifting, give a formula for it; if there is not a lifting, prove that it does not exist.

   (a) $f(z) = z^3$

   (b) $f(z) = z^4$.

4. Suppose $X = A \cup B$, where A and B are each homeomorphic to $S^1 \times S^1$, and $A \cap B$ is a simple closed curve C which is essential in A and essential in B. Compute $\pi_1(X)$. Be sure to say what the group is algebraically; don't just give a presentation of it. Hint: There is more than one way to look at X, hence more than one way to do the problem.

5. Let X be the space in Problem 4. Compute all the homology groups (with integer coefficients) of X. The same hint applies.

6. State the Eilenberg-Steenrod axioms for homology theory (with integer coefficients).

7. Suppose M is a closed, connected, orientable 5-manifold with $H_1(M) \cong \mathbf{Z}_6$ and $H_2(M) \cong \mathbf{Z} \oplus \mathbf{Z}_3$. Find all of the homology and cohomology groups of M. (Integer coefficients are assumed throughout.)

8. Let $\mathcal{A} = \{A_p, \partial_p^{\mathcal{A}}\}$, $\mathcal{B} = \{B_p, \partial_p^{\mathcal{B}}\}$, and $\mathcal{C} = \{C_p, \partial_p^{\mathcal{C}}\}$ be chain complexes. Let

   $$0 \to \mathcal{A} \xrightarrow{\phi} \mathcal{B} \xrightarrow{\psi} \mathcal{C} \to 0$$

   be an exact sequence of chain maps $\phi = \{\phi_p\}$ and $\psi = \{\psi_p\}$. Show how to define the connecting homomorphism $\partial_* : H_p(\mathcal{C}) \to H_{p-1}(\mathcal{A})$. You are NOT being asked to give the complete proof of the Zig-Zag Lemma. You are just being asked how you get from a representative cycle $c_p \in C_p$ to a representative cycle $a_{p-1} \in A_{p-1}$, why you can get there, and why a_{p-1} is a cycle. You do NOT need to show that the homology class of a_{p-1} is well-defined.