Geometric and Algebraic Topology
Comprehensive Exam – June 2018

Define your terminology and explain your notation. If you require a standard result, then state it before you use it; otherwise give clear and complete proofs of your claims.

Partial credit will be given for your work. Credit equivalent to five complete problems will guarantee a pass.

Geometric Topology

1. Suppose that G is a graph embedded in $\mathbb{R}^3$, with vertices V and edges E, and assume that the edges are straight line intervals. Replace each vertex $v \in V$ with a small ball $B_v \cong D^3$ centered at v, with the radii of the balls chosen so that each ball is disjoint from all of the other balls. Replace each edge $e \in E$ with a solid cylinder $C_e \cong D^2 \times [0,1]$ around e, with the radii of the solid cylinders small enough so that they are disjoint from the balls and other cylinders, except where a cylinder meets the balls at its endpoints. Let $H = \bigcup_{v\in V} B_v \cup \bigcup_{e\in E} C_e$ be the union of all of the balls and solid cylinders. Let S be the (necessarily orientable) boundary surface of H. Calculate the genus of S in terms of V and E.

2. Let D be the unit disk $D = \{z \in \mathbb{C} : |z| \leq 1\}$, and let $C = \{z \in \mathbb{C} : |z| = 1\}$ be its boundary. Let X be the real projective plane, obtained from D by identifying opposite points z and $-z$ on its boundary C. Note that under the identification, C becomes another circle $C' \subset X$. Now let Y be the topological space formed by gluing two copies of X along their respective circles C'. Calculate $\pi_1(Y)$.

3. Let $C = \{z \in \mathbb{C} : |z| = 1\}$ be the circle of radius one in the complex plane. Let A, $\widetilde{A}$, and B be copies of S^1. Let $p : \widetilde{A} \to A$ be the covering map given by $p(z) = z^2$. For each of the following maps $f : B \to A$, determine whether or not there is a lifting $g : B \to \widetilde{A}$. If there is such a lifting, give a formula for it. If there is not a lifting, prove that it does not exist.

   (a) $f(z) = z^3$

   (b) $f(z) = z^4$.

Algebraic Topology (Unless indicated otherwise all homology and cohomology groups have integer coefficients.)

4. Let X be the space obtained by identifying the edges of a 5-gon E using the pattern $a^{-1}bbaa$. Use cellular homology to calculate all the homology groups $H_n(X)$ with integer coefficients.

5. Let X be a space with $H_0(X) = \mathbb{Z}$, $H_1(X) = \mathbb{Z}_6 \oplus \mathbb{Z}_{15}$, $H_2(X) = \mathbb{Z}_5$, and $H_3(X) = 0$.

   (a) Compute $H^1(X)$, $H^2(X)$, and $H^3(X)$.

   (b) Compute $H^1(X;\mathbb{Z}_2)$, $H^2(X;\mathbb{Z}_2)$, and $H^3(X;\mathbb{Z}_2)$.

   (c) Compute $H^1(X;\mathbb{Z}_5)$, $H^2(X;\mathbb{Z}_5)$, and $H^3(X;\mathbb{Z}_5)$.

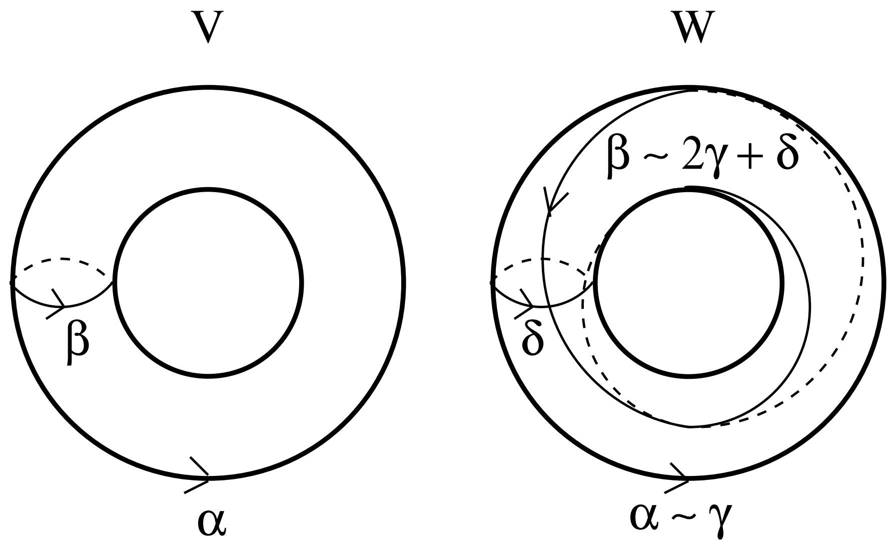
THERE IS ONE MORE PROBLEM ON THE NEXT PAGE

6. Recall that a solid torus is a space homeomorphic to $S^1 \times D^2$, where S^1 is the unit circle and D^2 is the closed unit disk in the complex plane. Its boundary is a torus $S^1 \times S^1$.

Suppose $X = V \cup W$, where V and W are solid tori and $V \cap W = T$ is their common boundary torus. Define the oriented simple closed curves α, β, γ and δ in T as follows. They are all non-trivial in $H_1(T)$. The curves α and β meet transversely in a single point, α generates $H_1(V)$, and β is trivial in $H_1(V)$. The curves γ and δ meet transversely in a single point, γ generates $H_1(W)$, and δ is trivial in $H_1(W)$.

Assume that in $H_1(T)$ we have $\alpha = \gamma$ and $\beta = 2\gamma + \delta$.

The figure below shows these curves as viewed on V and on W. The labels for the curves are next to the arrows on them. The symbol $\sim$ means "is homologous to in T."



Use the Mayer-Vietoris sequence to compute all the homology groups of X.

Hint: Remember that you need to know the homomorphisms in the sequence, not just the groups. In particular write down a formula for the homomorphism $H_1(T) \to H_1(V) \oplus H_1(W)$. It will have the form $(m, n) \mapsto (g(m, n), h(m, n))$.