Geometric and Algebraic Topology
Comprehensive Exam – January 2018

Define your terminology and explain your notation. If you require a standard result, then state it before you use it; otherwise give clear and complete proofs of your claims.

Partial credit will be given for your work. Credit equivalent to five complete problems will guarantee a pass.

Geometric Topology

1. Suppose that S is an orientable surface tiled by squares, such that at most four squares meet at each vertex of the tiling. Prove that S is either a sphere or a torus.

2. Recall that a topological space X is *semilocally simply connected* if every $x \in X$ has a neighborhood U such that the map $i_* : \pi_1(U, x) \to \pi_1(X, x)$ induced by the inclusion map $i : U \to X$ is trivial. Give an example of a topological space that is *not* semilocally simply connected, and briefly explain why it is not. *Hint: There are subsets of* $\mathbb{R}^2$*, formed as unions of circles, that are not semilocally simply connected.*

3. The Borsuk-Ulam theorem states that there does not exist a continuous antipode preserving map $f : S^n \to S^{n-1}$. Assuming the Borsuk-Ulam theorem, prove the following:

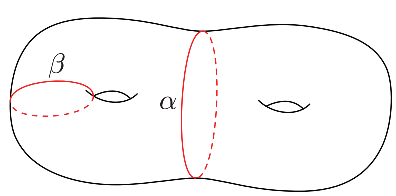
   (a) If $g : S^n \to \mathbb{R}^n$ is a continuous map such that $g(-x) = -g(x)$ for all $x \in S^n$, then there exists an $x \in S^n$ such that $g(x) = 0$.

   (b) If $h : S^n \to \mathbb{R}^n$ is a continuous map then there exists an $x \in S^n$ such that $h(x) = h(-x)$.

   (c) No subset of $\mathbb{R}^n$ is homeomorphic to S^n.

Algebraic Topology (Unless indicated otherwise all homology and cohomology groups have integer coefficients.)

4. Let F be the closed, connected, orientable surface of genus two.



   (a) Let X be a 2-complex obtained by attaching a disk D to F so that $D \cap F = \partial D$ is a simple closed curve α which separates F into two once-punctured tori. (A once-punctured torus is the result of removing the interior of a closed disk from a torus.) Use the Mayer-Vietoris sequence to compute the homology groups of X.

   (b) Let Y be a 2-complex obtained by attaching a disk D to F so that $D \cap F = \partial D$ is a simple closed curve β which does not separate F, i.e. the complement of ∂D in F is a twice-punctured torus. (A twice-punctured torus is the result of removing the interiors of two disjoint closed disks from a torus.) Use the Mayer-Vietoris sequence to compute the homology groups of Y.

THERE ARE MORE PROBLEMS ON THE NEXT PAGE

5. Regard the Klein bottle K as the space obtained from a rectangle E by gluing its edges in the pattern $abab^{-1}$. Let A be the circle in K which is the image of a, and let B be the circle in K which is the image of b. Let v be the point $A \cap B$.

   (a) Compute the relative homology groups $H_1(K, A)$ and $H_2(K, A)$.

   (b) Compute the relative homology groups $H_1(K, B)$ and $H_2(K, B)$.

   You may use the long exact sequence of a pair, or a direct computation using cellular homology, or a combination of these.

6. Let X be a space with $H_0(X) = \mathbb{Z}$, $H_1(X) = \mathbb{Z} \oplus \mathbb{Z}_6$, $H_2(X) = \mathbb{Z}_3$, and $H_3(X) = 0$.

   (a) Compute $H^1(X)$, $H^2(X)$, and $H^3(X)$.

   (b) Compute $H^1(X; \mathbb{Z}_2)$, $H^2(X; \mathbb{Z}_2)$, and $H^3(X; \mathbb{Z}_2)$.

   (c) Compute $H^1(X; \mathbb{Z}_3)$, $H^2(X; \mathbb{Z}_3)$, and $H^3(X; \mathbb{Z}_3)$.