Geometric and Algebraic Topology
Comprehensive Exam – August 2017

Define your terminology and explain your notation. If you require a standard result, then state it before you use it; otherwise give clear and complete proofs of your claims.

Partial credit will be given for your work. Credit equivalent to five complete problems will guarantee a pass.

Geometric Topology

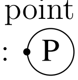
1. Let X and Y be path connected spaces. Let $X \vee Y$ denote the wedge sum of X and Y, i.e. the space obtained by identifying a single point in X with a single point in Y. Let S denote the 2-sphere and P the (real) projective plane.

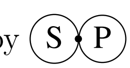
   (a) List (up to homeomorphism) all the finite sheeted connected covering spaces of $S \vee P$.

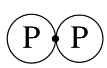
   (b) List (up to homeomorphism) all the finite sheeted connected covering spaces of $P \vee P$ having at most three sheets.

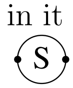
   (c) Describe the universal covering space of $P \vee P$.

   <u>Hints</u>: Denote the pair consisting of the sphere and one point in it by a circle with a dot on it and an S inside it, as follows: 

   Denote the pair consisting of the projective plane and one point in it by a circle with a dot on it and a P inside it, as follows: 

   Then $S \vee P$ is represented by 

   and $P \vee P$ is represented by 

   Denote the pair consisting of the sphere and two points in it by a circle with two dots on it and an S inside it, as follows: 

   Draw pictures of the covering spaces by hooking these various circles together at the dots.

2. Find the fundamental groups of the following spaces. In each case they can be built up from cyclic groups by free products and direct products.

   (a) The space obtained from two copies of the torus $S^1 \times S^1$ by identifying the simple closed curve $S^1 \times \{1\}$ on the first copy of the torus with the simple closed curve $S^1 \times \{1\}$ on the second copy of the torus. (Regard S^1 as the set of complex numbers with unit norm.)

   (b) The space obtained from a torus $S^1 \times S^1$ by joining two distinct points a and b on it by an arc which meets the torus only in its two endpoints a and b.

3. Let S be the surface obtained by identifying the sides of a 12-gon in the pattern $abcd^{-1}a^{-1}ef^{-1}b^{-1}fe^{-1}c^{-1}d$.

   (a) Compute the Euler characteristic of S.

   (b) Determine whether the surface is orientable or non-orientable.

   (c) Express S as a connected sum of tori or as a connected sum of projective planes. (On this part of the problem you do not need to show your work; just write down the final answer.)

Algebraic Topology (All homology and cohomology groups have $\mathbb{Z}$ coefficients.)

4. Let X be a space formed from a cube $\{(x, y, z) \in \mathbb{R} \mid -1 \leq x, y, z \leq 1\}$ by identifying opposite sides via the following bijections:

$$(x, y, -1) \leftrightarrow (-x, y, 1)$$

$$(x, -1, z) \leftrightarrow (x, 1, z)$$

$$(-1, y, z) \leftrightarrow (1, y, z).$$

   In other words, the faces parallel to the xz and yz planes are glued by the identity map, while the faces parallel to the xy plane are glued with a reflection. Use cellular homology to calculate the homology groups $H_n(X)$ for $n \geq 0$.

5. Recall that the degree $\deg f$ of a map $f : S^n \to S^n$ is the number d such that for the induced map $f_* : H_n(S^n) \to H_n(S^n)$ is given by $f_*(\alpha) = d\alpha$.

   (a) Show that the antipodal map $-id : S^n \to S^n$ has degree $(-1)^{n+1}$.

   (b) Show that any map $f : S^n \to S^n$ with no fixed points is homotopic to the antipodal map.

   (c) Deduce that any map $f : S^{2n} \to S^{2n}$ has a fixed point.

6. The suspension ΣX of a space X is the space obtained from $X \times [-1.1]$ by identifying $X \times \{-1\}$ to a point (the "south pole") and $X \times \{1\}$ to a point (the "north pole"). Thus ΣX is the union of two copies of a cone CX on X along a common copy of X. For example, if X is S^1, then CX is a disk and ΣX is the union of two disks along their boundaries, so is homeomorphic to S^2. State and prove a result relating the homology groups of X and ΣX. *Hint: use the Mayer-Vietoris sequence.*