Geometric and Algebraic Topology

Comprehensive Exam – January 2017

Define your terminology and explain your notation. If you require a standard result, then state it before you use it; otherwise give clear and complete proofs of your claims.

Partial credit will be given for your work. Credit equivalent to five complete problems will guarantee a pass.

Geometric Topology

1. Show that $T^2 \# \mathbb{R}P^2$ is homeomorphic to $K \# \mathbb{R}P^2$, where T^2 is the torus, K is the Klein bottle, and $\mathbb{R}P^2$ is the real projective plane. You may use either cut and paste arguments, or apply a general theorem.

2. Let X be a topological space, and $f : [0,1] \to X$ a path in X. Let $\cdot$ denote concatenation of paths, so
$$(f \cdot g)(t) = \begin{cases} f(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ g(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$
Let $\bar{f}$ denote the reverse of the path f, so $\bar{f}(t) = f(1-t)$.
Show that $f \cdot \bar{f}$ is homotopic to a constant path, via a homotopy $F : [0,1] \times [0,1] \to X$ for which $F(0,s) = f(0) = \bar{f}(1) = F(1,s)$ for all $s \in [0,1]$.

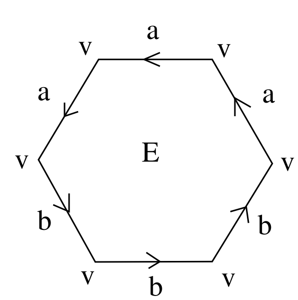
3. (a) Show that there is a circle C which is a deformation retract of the Möbius strip. You may either give a formula, or describe the deformation retract geometrically.
   (b) Prove that the boundary of the Möbius strip is not a retract of the Möbius strip.

Algebraic Topology (Unless indicated otherwise all homology and cohomology groups have integer coefficients.)

4. M is a closed, connected, orientable n-manifold, $n \ge 3$. Prove that if $H_1(M)$ is finite, then $H_{n-1}(M) = 0$.

5. X and Y are spaces with $H_0(X) = \mathbb{Z}$, $H_1(X) = \mathbb{Z}_4$, $H_0(Y) = \mathbb{Z}$, $H_1(Y) = \mathbb{Z}_6$, and $H_p(X) = H_p(Y) = 0$ for all $p \ge 2$. Compute $H_p(X \times Y)$ for all $p \ge 0$.

6. Let X be the CW-complex indicated below. It has one 0-cell v, two 1-cells a and b, and one 2-cell E.



Using cellular homology compute $H_p(X)$ for all $p \ge 0$. In particular compute all the boundary maps $\partial_p : C_p(X) \to C_{p-1}(X)$, give bases for all the $Z_p(X)$ and $B_p(X)$, and use these to compute $H_p(X)$. (Hint: You may sometimes want to change bases.)