Geometric and Algebraic Topology
Comprehensive Exam – August 2015

Define your terminology and explain your notation. If you require a standard result, then state it before you use it; otherwise give clear and complete proofs of your claims.

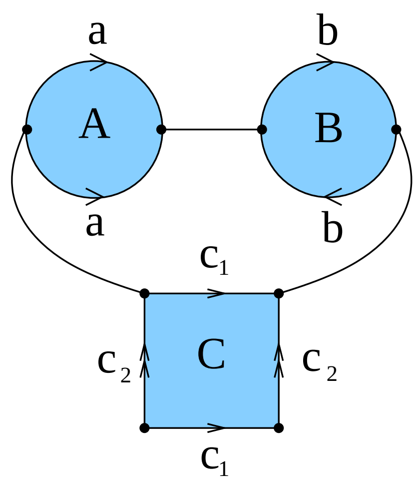
There are six problems, three each in geometric and algebraic topology.

Partial credit will be given for your work. Credit equivalent to five complete problems will guarantee a pass.

Geometric Topology

1. Let X be a space with $\pi_1(X) = \mathbb{Z} * \mathbb{Z}$. Let Y be a space with $\pi_1(Y) = \mathbb{Z} \times \mathbb{Z}$. Prove that there is no space W which is a finite sheeted covering space of both X and Y. Assume that all spaces are path connected, locally path connected, and semi-locally simply connected. You may quote standard facts from group theory.

2. Let X be the cell complex in the figure below. Oriented edges with the same labels are identified as shown. Find the fundamental group of X. Express your final answer using cyclic groups and free and/or direct products.



3. Let P be the (real) projective plane. Suppose $\mathcal{T}$ is a simplicial triangulation of P. Recall that this implies that each simplex has distinct vertices and is uniquely determined by its vertices. The *order* of a vertex of $\mathcal{T}$ is the number of 1-simplices meeting it. Prove that if every vertex has the same order n, then $n = 5$. Determine the number of vertices, the number of edges, and the number of triangles in this case. Hint: Use the Euler characteristic to list all the numerical possibilities for n and the number of vertices. Then eliminate geometrically impossible cases. You do not have to prove that a triangulation with $n = 5$ exists.

Algebraic Topology

If no other coefficient group G is specified, then assume that $G = \mathbb{Z}$.

1. Suppose that $\mathcal{A} = \{A_n, \partial_n^A\}$, $\mathcal{B} = \{B_n, \partial_n^B\}$ and $\mathcal{C} = \{C_n, \partial_n^C\}$ are chain complexes, with maps $\phi_n : A_n \to B_n$ and $\psi_n : B_n \to C_n$ which make the following diagram commute, and have exact rows.

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r, "\phi_n"] \arrow[d, "\partial_n^A"] & B_n \arrow[r, "\psi_n"] \arrow[d, "\partial_n^B"] & C_n \arrow[r] \arrow[d, "\partial_n^C"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "\phi_{n-1}"] & B_{n-1} \arrow[r, "\psi_{n-1}"] & C_{n-1} \arrow[r] & 0
\end{tikzcd}$$

   Suppose that a cycle $c_n \in C_n$ represents an element $\gamma \in H_n(\mathcal{C})$. Show how to find a representative cycle $a_{n-1} \in H_{n-1}(\mathcal{A})$ of the image $\partial_*(\gamma)$ of the connecting homomorphism $\partial_* : H_n(\mathcal{C}) \to H_{n-1}(\mathcal{A})$. Explain why a_{n-1} is a cycle, but do not show that the homology class of a_{n-1} is well defined.

2. Let K be the interior of a closed solid torus $T \cong D^2 \times S^1$ embedded in S^3. So $K = \mathring{T}$, and $T \subset S^3$. By using Mayer-Vietoris, or by any other method, show that

$$H_1(S^3 \setminus K; \mathbb{Z}) \cong \mathbb{Z}.$$

   Note that K may be knotted, so $S^3 \setminus K$ is not necessarily homeomorphic to another solid torus.

3. Suppose that X is a closed, connected, orientable 4-manifold, with Error $H_1(X;\mathbb{Z}) = \mathbb{Z}_4$ and $H_2(X;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_6$. Calculate all other homology and cohomology groups with $\mathbb{Z}$ coefficients.