Geometric and Algebraic Topology
Comprehensive Exam – June 2015

Define your terminology and explain your notation. If you require a standard result, then state it before you use it; otherwise give clear and complete proofs of your claims.

There are six problems, three each in geometric and algebraic topology.

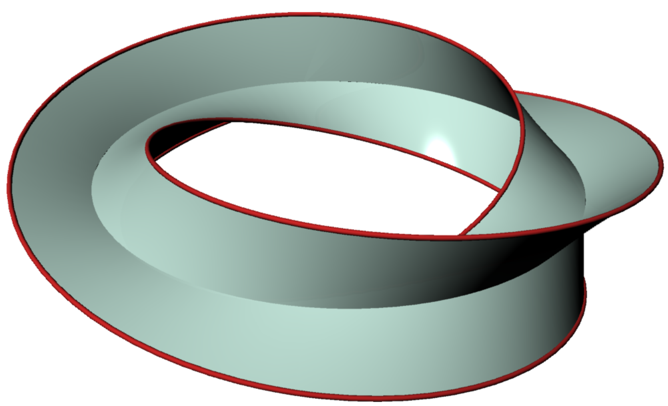
Partial credit will be given for your work. Credit equivalent to five complete problems will guarantee a pass.

Geometric Topology

1. Let $\mathcal{C}$ be a cellulation of S^2 having $T \geq 0$ triangles and $Q \geq 0$ quadrilaterals. Assume that the degree of every vertex of $\mathcal{C}$ is 3. List all of the possibilities for the pair (T, Q). Prove your answer.

2. Let Y be a graph in $\mathbb{R}^2$ consisting of three edges connected to a single vertex to form the shape of a "Y". Let X be obtained by sweeping Y around a circle in $\mathbb{R}^3$, rotating it by $4\pi/3$ before connecting up. See the figure. In other words, $X \cong (Y \times [0,1])/\sim$, where $(y, 0) \sim (\rho(y), 1)$, for $\rho : \mathbb{C} \to \mathbb{C}$ given by $\rho(z) = ze^{4\pi i/3}$.

   Note that the three ends of Y trace out a knot $K = (\partial Y \times [0,1])/\sim$ in $\mathbb{R}^3$. Let Z be the quotient space obtained from X by collapsing K to a point. Calculate the fundamental group of Z.



3. Determine all covering spaces (up to isomorphism) of the circle. Prove that your list is complete.

Algebraic Topology

If no other coefficient group G is specified, then assume that $G = \mathbb{Z}$.

4. Let X be a space with $H_0(X) = \mathbb{Z}$, $H_1(X) = \mathbb{Z} \oplus \mathbb{Z}_6$, $H_2(X) = \mathbb{Z}_4$, and $H_p(X) = 0$ for all $p > 2$.

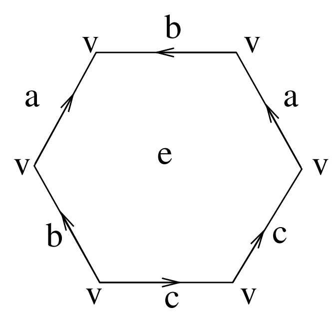
   You may assume the standard facts about the universal coefficient theorems and Ext and Tor.

   (a) Compute $H^p(X)$ for all $p > 0$.

   (b) Compute $H^p(X; \mathbb{Z}_2)$ for all $p > 0$.

   (c) Compute $H_p(X; \mathbb{Z}_2)$ for all $p > 0$.

5. Let X be the CW-complex indicated below. It has one 0-cell v, three 1-cells a, b, c, and one 2-cell e.



   (a) Compute $H_1(X)$ and $H_2(X)$ using cellular homology.

   (b) Compute $H^1(X)$ and $H^2(X)$ using cellular cohomology.

   Do NOT compute $H_0(X)$ and $H^0(X)$. Do NOT use the universal coefficient theorem.

   Hints on (b): Use a^* to denote the 1-cochain such that $\langle a^*, a\rangle = 1$, $\langle a^*, b\rangle = 0$, and $\langle a^*, c\rangle = 0$. Use similar definitions for b^*, c^*, for e^*, and for v^*. These form bases for their respective cochain groups. Recall that the coboundary operator δ is defined by $\langle \delta w^p, y_{p+1}\rangle = \langle w^p, \partial y_{p+1}\rangle$. (The notation $\langle x^q, y_q\rangle$ means the evaluation of the q-cochain x^q on the q-chain y_q.)

6. Let $X = T \cup D$, where T is a torus, D is a closed disk, $A = T \cap D = \partial D$, and A is a circle which represents one of the two basis elements for $H_1(T) = \mathbb{Z} \oplus \mathbb{Z}$. Use the Mayer-Vietoris sequence to compute $H_1(X)$ and $H_2(X)$.